5. F. B. HILDEBRAND, *Methods of Applied Mathematics.* Englewood Cliffs: Prentice-Hall, 1952.

6. R. WEINSTOCK, *Calculus of Variations.* New York: McGraw-Hill, 1952.

Problems

8–1. Reconsider Example 8–2. Find the first- and second-order approximate solutions in terms of orthogonal circular functions by using (a) Eq. (8–23), (b) Eq. (8–26). (c) Compare the complexity of the algebra involved in both cases. (d) Show that the value of the unknown parameters is independent of the order of approximation. (e) Find the higher-order approximations by inspecting the first two approximations. (f) Show that part (e) is equivalent to the expansion of the exact solution $\cosh \mu\xi/\cosh \mu$ into a Fourier cosine series. (g) Compare the accuracy of the results obtained using circular profiles with that of results obtained using polynomials, given in Example 8–2.

8–2. In a small steam boiler, the rims connecting the furnace to the body are made up of thin rods (Fig. 8–23). (a) Show that the steady temperature problem associated with the rims can be made identical to a variational problem. (b) Construct approximate temperature profiles in terms of polynomials. (c) Find the temperature of the rims by using the polynomials of part (b).

8–3. Condensing steam at temperature T_0 flows through a thick-walled pipe (Fig. 8–24). The inner and outer radii of the pipe are R_i, R_o, respectively. The heat transfer coefficient between the outer surface of the pipe and the ambient is h, and the ambient temperature is T_∞.* Under steady conditions, (a) find the functional asso-

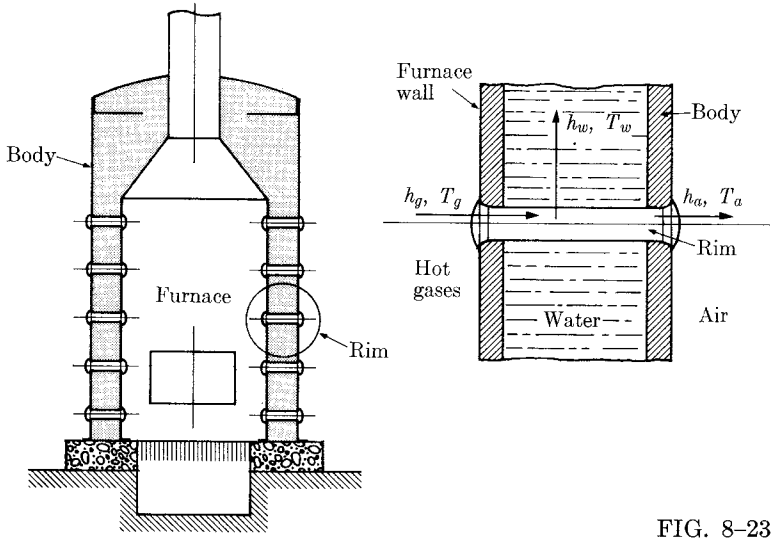


FIG. 8–23

* This problem is the steady case of Problem 7–24.

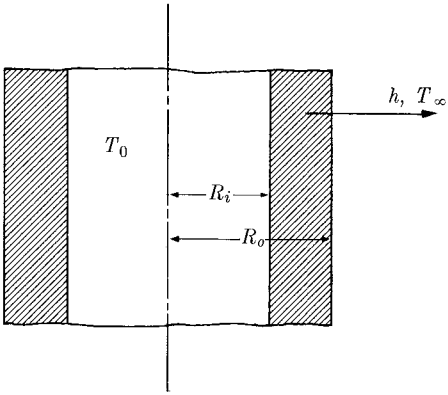


FIG. 8–24

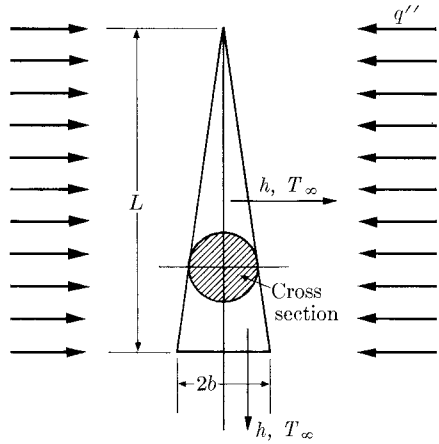


FIG. 8–25

ciated with the temperature of the pipe walls; (b) construct approximate temperature profiles in terms of polynomials; (c) obtain the temperature of the pipe walls by using the polynomials of part (b).

8–4. Reconsider Example 8–3. (a) Find the third-order approximation of the temperature of the disks in terms of polynomials. (b) Compare the centerline temperature of the third-order approximation to that of the second-order approximation as a function of μ.

8–5. A flying object of solid conical shape receives the peripheral radiant heat q'' (Fig. 8–25). The ambient temperature is T_∞, and the heat transfer coefficient is h. The bottom radius and the height of the cone are b and L, respectively, such that $b/L \ll 1$. (a) Find the functional associated with the steady temperature of the object. (b) Obtain an approximate temperature distribution in terms of polynomials.

8–6. Reconsider Example 8–4. (a) Find the first two approximations of the steady heater temperature for a large heat transfer coefficient by using the Ritz and Kantorovich profiles expressed in terms of circular functions. (b) Compare the accuracy of these circular profiles with that of profiles based on polynomials, using the first approximations only. (See Examples 8–5 and 8–6.)

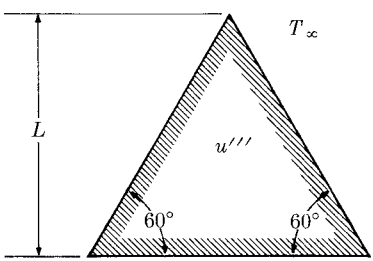


FIG. 8–26

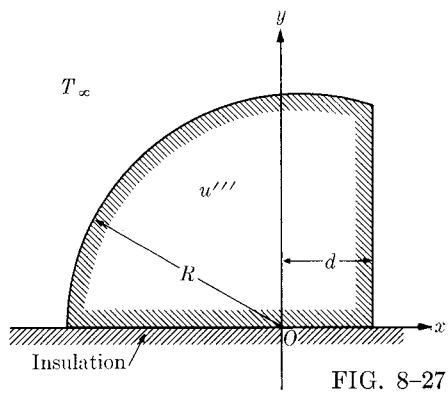


FIG. 8–27

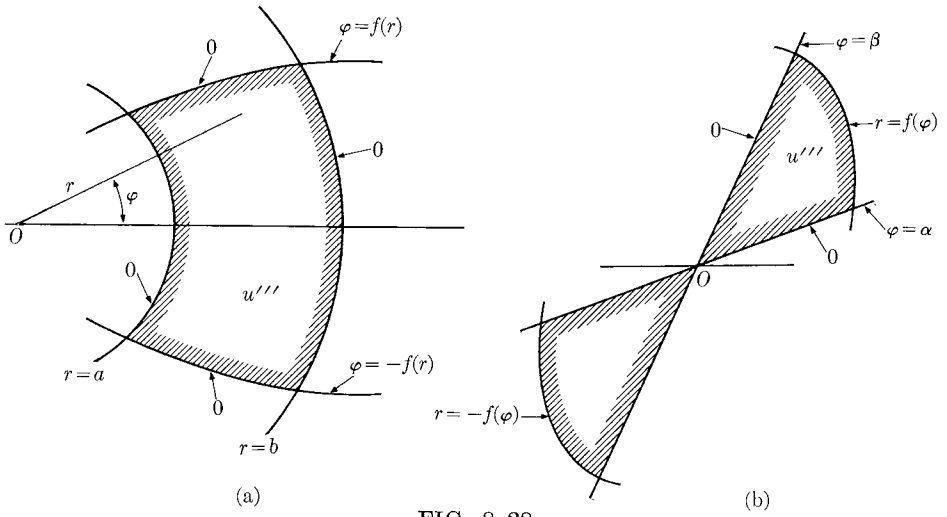


FIG. 8–28

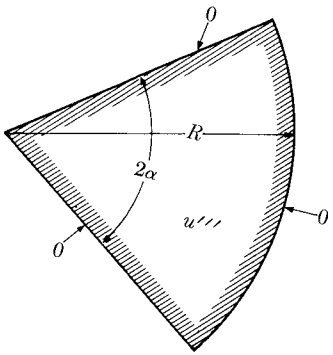


FIG. 8–29

8–7. The internal energy u''' is generated uniformly in a solid triangular rod (Fig. 8–26). The heat transfer coefficient is large. Calculate the steady temperature of this rod, using Ritz and Kantorovich profiles expressed in terms of polynomials.

8–8. The internal energy u''' is generated uniformly in a rod whose cross section is shown in Fig. 8–27. One surface of the rod is insulated while the others transfer heat to the ambient with a large heat transfer coefficient. Calculate the steady temperature of this rod, using (a) cartesian Ritz and Kantorovich profiles expressed in terms of polynomials, (b) polar Ritz profiles.

8–9. Generalize the procedure given in Section 8–9 to polar geometry, as follows. (a) Using Fig. 8–28, find the formulas equivalent to those of Eqs. (8–100) and (8–101). (b) Apply the result of Fig. 8–28 to the problem given by Fig. 8–29.

8–10. Write a steady Kantorovich profile for the problems shown in Fig. 8–30.

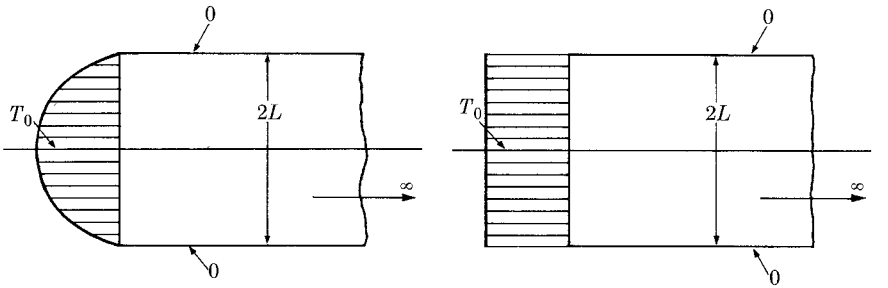


FIG. 8–30

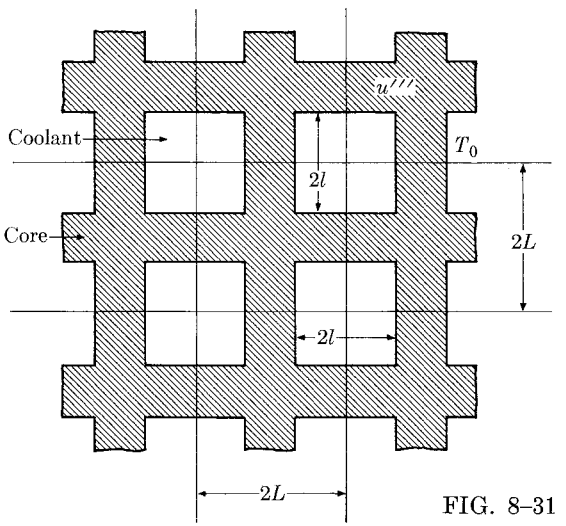


FIG. 8–31

8–11. The internal energy u''' generated uniformly in the core of a nuclear reactor is removed by a coolant flowing at temperature T_0 through the core (Fig. 8–31). Find the first two approximations of the core temperature under steady conditions, assuming (a) a large heat transfer coefficient, (b) a moderate heat transfer coefficient h between the core and the coolant.

8–12. Reconsider Example 2–2. Find the first two approximations of the temperature of the fuel plates, assuming (a) a large heat transfer coefficient, (b) a moderate heat transfer coefficient h between the plates and the coolant.

8–13. Re-solve Problem 5–11.

8–14. Re-solve Problem 5–12.

8–15. Re-solve Example 5–7.

8–16. Re-solve Problem 5–18.

8–17. Re-solve Problem 5–22.

8–18. Re-solve Problem 5–24.

8–19. Find the first- and second-order profiles of the *unsteady* problem associated with Example 2–1.

8–20. Find the first-order profile of the *unsteady* problem associated with Problem 8–7.

DIFFERENCE FORMULATION. NUMERICAL AND GRAPHICAL SOLUTIONS

Except for a few trivial cases, so far we have considered only single-domain problems of regular geometry. When the boundaries are irregular, that is, not parallel to coordinates of a system, or when we are dealing with a multidomain problem, there is no exact solution available by the known analytical methods. We learned in previous chapters that many problems of this type are solvable by the approximate integral and variational techniques; however, even these techniques are not flexible enough to handle all irregular geometry and/or multidomain problems. The difference method is the only procedure suitable particularly to these problems. Hence we devote this chapter to the difference formulation and its solution by numerical and graphical means. The analog solution of the same formulation will be the subject matter of Chapter 10. Since the difference formulation of steady problems requires solution techniques different from those used for unsteady problems, we shall investigate steady and unsteady problems separately.

9–1. Difference Formulation of Steady Problems

First we shall consider two-dimensional cartesian geometry; then we shall extend the results to three-dimensional cartesian geometry, and to cylindrical and spherical geometries. We illustrate two-dimensional cartesian geometry in terms of Example 2–1 (Fig. 9–1).

Let us follow again the five steps of the formulation (see Section 2–9) and rearrange them for the difference formulation.

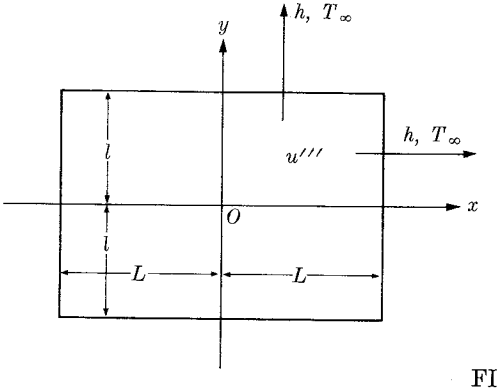
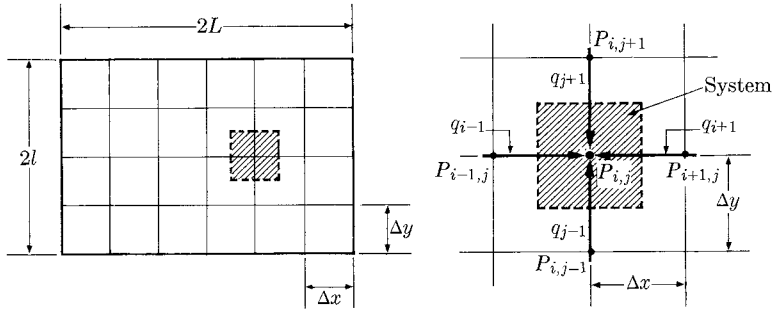


FIG. 9–1



(a) (b)

FIG. 9–2

(i) *Define a system* by subdividing the electric heater into a number of squares of small but finite size, forming the so-called *network* (Fig. 9–2a). Here a square network is chosen for convenience only. The geometry or other factors may indicate the use of rectangular, triangular, or hexagonal networks instead (see Section 9–4). For example, suppose that the geometry of the electric heater of Fig. 9–2 did not permit a square network of desired size. Then the use of a rectangular network or a suitable combination of square and rectangular networks would become necessary. The intersections of networks are called the *nodal points* P, and are identified by two subscripts, say i and j, to indicate the row and the column of the point, respectively. Nodal points are classified according to their location as *inner* and *boundary* nodal points. The system for a typical inner nodal point is shown in Fig. 9–2(b). The systems for boundary nodal points are considered in step (v) of the formulation.

(ii) *State the general law* in terms of the system defined by the preceding step. The first law of thermodynamics applied to the system of Fig. 9–2(b) gives

$$0 = q_{i-1}(\Delta y \cdot 1) + q_{j-1}(\Delta x \cdot 1) + q_{i+1}(\Delta y \cdot 1) + q_{j+1}(\Delta x \cdot 1) + u'''(\Delta x\, \Delta y \cdot 1), \qquad (9\text{–}1)$$

where 1 denotes the unit thickness. The directions of heat fluxes are arbitrarily selected toward the nodal point for algebraic convenience. Thus all signs appearing in Eq. (9–1) are made positive.

(iii) *State the particular law* in terms of the preceding steps. Relating q_{i-1}, q_{j-1}, q_{i+1}, and q_{j+1} to the temperature by Fourier's law of conduction we have

$$q_{i-1} = k\frac{T_{i-1,j} - T_{i,j}}{\Delta x}, \qquad q_{j-1} = k\frac{T_{i,j-1} - T_{i,j}}{\Delta y},$$
$$q_{i+1} = k\frac{T_{i+1,j} - T_{i,j}}{\Delta x}, \qquad q_{j+1} = k\frac{T_{i,j+1} - T_{i,j}}{\Delta y}, \qquad (9\text{–}2)$$

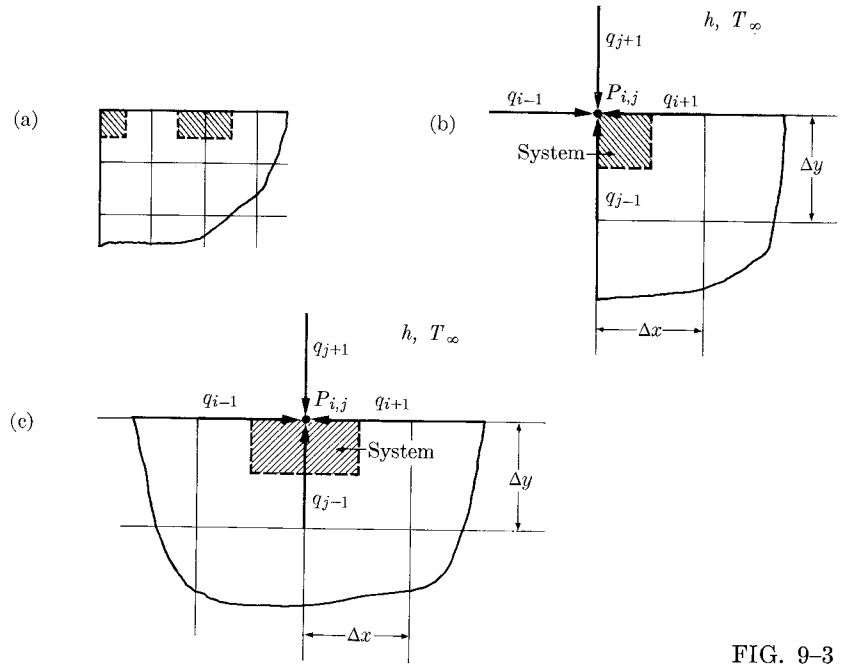


FIG. 9–3

where T denotes the temperature, with appropriate subscripts corresponding to the nodal points.

(iv) *Obtain the governing difference equation* by inserting Eq. (9–2) into Eq. (9–1). Noting that $\Delta x = \Delta y$ (say Δl), and assuming the material of the heater to be homogeneous and isotropic, we find that

$$T_{i-1,j} + T_{i,j-1} + T_{i+1,j} + T_{i,j+1} - 4T_{i,j} + u'''(\Delta l)^2/k = 0. \qquad (9\text{–}3)$$

Eq. (9–3) applies equally to *all* inner nodal points of Fig. 9–2(a).

(v) *Write the boundary conditions* in terms of typical boundary nodal points and their associated systems, shown in Fig. 9–3(a) and elaborated in Fig. 9–3(b, c). Note that the difference form of boundary conditions, unlike their differential form, is related to a volume. Thus the statement of this form of boundary conditions requires that the first four steps of formulation be applied to the boundary systems shown in Fig. 9–3(b, c).

The first law of thermodynamics applied to the system of Fig. 9–3(b) gives

$$0 = q_{i-1}\left(\frac{\Delta y}{2}\cdot 1\right) + q_{j-1}\left(\frac{\Delta x}{2}\cdot 1\right) + q_{i+1}\left(\frac{\Delta y}{2}\cdot 1\right) + q_{j+1}\left(\frac{\Delta x}{2}\cdot 1\right) + u'''\left(\frac{\Delta x}{2}\,\frac{\Delta y}{2}\cdot 1\right). \qquad (9\text{–}4)$$

Relating q_{i-1} and q_{j+1} to the temperature by the definition of h,

$$q_{i-1} = q_{j+1} = h(T_\infty - T_{i,j}),$$

and q_{j-1} and q_{i+1} to the temperature by Eq. (9–2), and noting that

$$\Delta x = \Delta y = \Delta l,$$

we have

$$\frac{1}{2}\left(T_{i,j-1} + T_{i+1,j}\right) - \left(1 + \frac{h\,\Delta l}{k}\right) T_{i,j} + \frac{h\,\Delta l}{k}\,T_\infty + \frac{u'''(\Delta l)^2}{4k} = 0. \tag{9–5}$$

Equation (9–5) applies to the corners of the heater. Following the foregoing procedure in terms of Fig. 9–3(c) we find that the difference equation for the remaining boundaries is

$$\frac{1}{2}\left(T_{i-1,j} + T_{i+1,j}\right) + T_{i,j-1} - \left(2 + \frac{h\,\Delta l}{k}\right) T_{i,j} + \frac{h\,\Delta l}{k}\,T_\infty + \frac{u'''(\Delta l)^2}{2k} = 0. \tag{9–6}$$

Thus the formulation of the problem is reduced to a set of simultaneous algebraic equations composed of three typical equations given by Eqs. (9–3), (9–5), and (9–6). (How many equations and unknowns?) The numerical solution of these algebraic equations will be discussed in Section 9–7.

9–2. Relation Between Difference and Differential Formulations

In Section 2–9 we learned that problems may be formulated according to the physical or the mathematical approach. Throughout the text, we have preferred to use the physical approach, and in the preceding section we obtained the difference formulation of steady problems by that same approach. Now, using the mathematical approach, we shall show the relation between the difference and the differential formulations. For this purpose, consider again the differential formulation of Example 2–1. From Chapter 2, we have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{u'''}{k} = 0. \tag{2–128}$$

Since Eq. (2–128) applies locally, it may be written for an inner nodal point $P_{i,j}$ as follows:

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{u'''}{k}\right)_{P_{i,j}} = 0. \tag{9–7}$$

Now, replacing the partial derivatives by the equivalent difference forms yields

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{P_{i,j}} = \left(\frac{\Delta_x^2 T}{\Delta x^2}\right)_{P_{i,j}} = \left.\frac{\Delta_x(\Delta_x T/\Delta x)}{\Delta x}\right|_{P_{i,j}}, \tag{9–8}$$

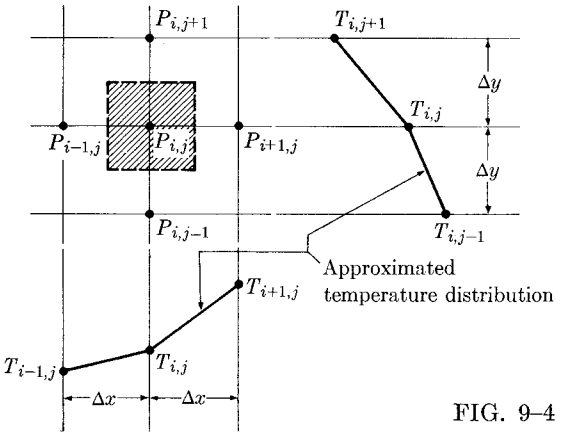


FIG. 9–4

where Δ_x denotes the difference in the direction of x. The numerator of the last expression in Eq. (9–8) is simply the difference in $\Delta_x T/\Delta x$ at point $P_{i,j}$ in the direction of x. It follows from Fig. 9–4 that

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{P_{i,j}} = \frac{1}{\Delta x}\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x} - \frac{T_{i,j} - T_{i-1,j}}{\Delta x}\right)$$

$$= \frac{1}{(\Delta x)^2}(T_{i+1,j} + T_{i-1,j} - 2T_{i,j}). \tag{9–9}$$

Similarly,

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{P_{i,j}} = \frac{1}{(\Delta y)^2}(T_{i,j+1} + T_{i,j-1} - 2T_{i,j}). \tag{9–10}$$

Introducing Eqs. (9–9) and (9–10) into Eq. (9–7), and again assuming a square network so that $\Delta x = \Delta y = \Delta l$, we obtain Eq. (9–3), as expected. [Obtain the difference form of the boundary conditions from their differential form. Is the result identical to Eqs. (9–5) and (9–6)? Why?]

9–3. Error in Difference Formulation

We now investigate the error involved in the difference formulation of steady problems, again in terms of Example 2–1. Recalling the expression of the one-dimensional Taylor series in the neighborhood of a point, say ζ_0,

$$T(\zeta) = T(\zeta_0) + (\zeta - \zeta_0)\left(\frac{dT}{d\zeta}\right)_{\zeta=\zeta_0} + \frac{1}{2!}(\zeta - \zeta_0)^2\left(\frac{d^2T}{d\zeta^2}\right)_{\zeta=\zeta_0}$$

$$+ \frac{1}{3!}(\zeta - \zeta_0)^3\left(\frac{d^3T}{d\zeta^3}\right)_{\zeta=\zeta_0} + \frac{1}{4!}(\zeta - \zeta_0)^4\left(\frac{d^4T}{d\zeta^4}\right)_{\zeta=\zeta_0} + \cdots,$$

we may express $T_{i-1,j}$, $T_{i+1,j}$, $T_{i,j-1}$, and $T_{i,j+1}$ in terms of $T_{i,j}$ as follows:

$$T_{i-1,j} = T_{i,j} - \Delta x \left(\frac{\partial T}{\partial x}\right)_{P_{i,j}} + \frac{1}{2!}(\Delta x)^2 \left(\frac{\partial^2 T}{\partial x^2}\right)_{P_{i,j}} - \frac{1}{3!}(\Delta x)^3 \left(\frac{\partial^3 T}{\partial x^3}\right)_{P_{i,j}} + \frac{1}{4!}(\Delta x)^4 \left(\frac{\partial^4 T}{\partial x^4}\right)_{P_{i,j}} - \cdots,$$

$$T_{i+1,j} = T_{i,j} + \Delta x \left(\frac{\partial T}{\partial x}\right)_{P_{i,j}} + \frac{1}{2!}(\Delta x)^2 \left(\frac{\partial^2 T}{\partial x^2}\right)_{P_{i,j}} + \frac{1}{3!}(\Delta x)^3 \left(\frac{\partial^3 T}{\partial x^3}\right)_{P_{i,j}} + \frac{1}{4!}(\Delta x)^4 \left(\frac{\partial^4 T}{\partial x^4}\right)_{P_{i,j}} + \cdots,$$

$$T_{i,j-1} = T_{i,j} - \Delta y \left(\frac{\partial T}{\partial y}\right)_{P_{i,j}} + \frac{1}{2!}(\Delta y)^2 \left(\frac{\partial^2 T}{\partial y^2}\right)_{P_{i,j}} - \frac{1}{3!}(\Delta y)^3 \left(\frac{\partial^3 T}{\partial y^3}\right)_{P_{i,j}} + \frac{1}{4!}(\Delta y)^4 \left(\frac{\partial^4 T}{\partial y^4}\right)_{P_{i,j}} - \cdots,$$

$$T_{i,j+1} = T_{i,j} + \Delta y \left(\frac{\partial T}{\partial y}\right)_{P_{i,j}} + \frac{1}{2!}(\Delta y)^2 \left(\frac{\partial^2 T}{\partial y^2}\right)_{P_{i,j}} + \frac{1}{3!}(\Delta y)^3 \left(\frac{\partial^3 T}{\partial y^3}\right)_{P_{i,j}} + \frac{1}{4!}(\Delta y)^4 \left(\frac{\partial^4 T}{\partial y^4}\right)_{P_{i,j}} + \cdots$$

Adding these expressions together, we have for a square network

$$T_{i-1,j} + T_{i,j-1} + T_{i+1,j} + T_{i,j+1} - 4T_{i,j} = (\Delta l)^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)_{P_{i,j}} + \mathrm{O}\,[(\Delta l)^4], \qquad (9\text{–}11)$$

where all terms containing fourth or higher powers of Δl are included in $\mathrm{O}[(\Delta l)^4]$. Combining Eq. (9–7) and (9–11), we obtain

$$T_{i-1,j} + T_{i,j-1} + T_{i+1,j} + T_{i,j+1} - 4T_{i,j} + u'''(\Delta l)^2/k = \mathrm{O}[(\Delta l)^4]. \qquad (9\text{–}12)$$

Neglecting $\mathrm{O}[(\Delta l)^4]$, which becomes smaller as $\Delta l \to 0$, we obtain the difference formulation given by Eq. (9–3). This formulation is therefore an approximation to the differential formulation only when $(\Delta l)^4$ is small. $\mathrm{O}[(\Delta l)^4]$ is called the *truncation error* of Eq. (9–3). Clearly, this error is minimized by selecting as small a network as is practicable. On the other hand, since the smaller the network, the larger the number of nodal points and associated difference equa-

tions, from the viewpoint of the algebraic simplicity of the solution we would prefer to select as coarse a network as possible. The best network size is therefore a compromise, and there is no rule other than insight and experience for choosing the optimum size in a given situation. Until much practice has been gained, the only way to judge the errors introduced by too coarse a network is to check the change in temperature with successively finer networks.

9–4. Finer, Graded, Triangular, and Hexagonal Networks

To determine the accuracy of a given difference formulation, it is necessary to formulate the problem for at least two different sizes of network. Successively *finer networks* may be chosen, often each to have squares one-quarter as big as those of the previous network.

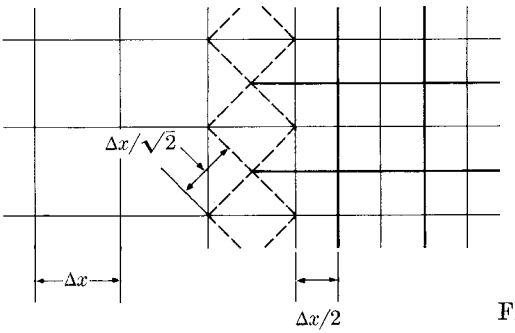
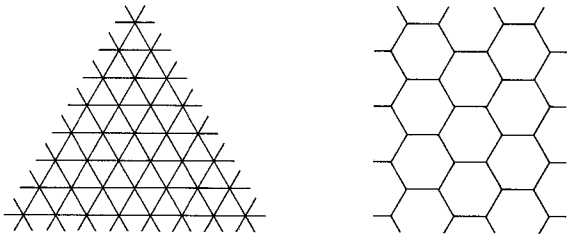


FIG. 9–5

In the difference formulation of some problems a fine network may be needed only in some part of the integration domain. For example, we may wish to have a greater accuracy in one part of the solution; or for the same accuracy we may be forced to a smaller network in one part of the field than in the remainder because the temperature may be varying more rapidly in that part. It is possible, of course, to extend the fine network to the entire domain, but this would entail unnecessary expenditure of time and effort in the solution. Therefore, it is convenient to know how to use networks whose size differs in different parts of the field. A network which exhibits this feature is said to be a *graded network*. We may illustrate the point on a two-dimensional network by halving one part of the network while keeping the remainder unchanged (Fig. 9–5). (Note that an intermediate network of side $\Delta x/\sqrt{2}$ is required to join the coarse and fine networks and that the intermediate network is oriented at 45° to the other two networks.)

It is geometrically possible to cover an integration domain by a uniform network made of squares, or *equilateral triangles* (Fig. 9–6a), or *equiangular hexagons* (Fig. 9–6b). So far we have dealt with square networks. On rare occasions the triangular or hexagonal networks are also used. For example, if



(a) (b)

FIG. 9–6

the shape of the boundary is such that a triangular or hexagonal network exactly fits to the field, it is more convenient as well as more accurate to employ these networks.*

9–5. Cylindrical and Spherical Geometries

We now consider the difference formulation of cylindrical problems. Since no additional concept is needed for spherical problems, we leave these to the reader as an exercise. Let us illustrate the cylindrical case in terms of the electric heater of Problem 4–20 (Fig. 9–7). In accordance with the physical approach, we take the appropriate systems as indicated on one-quarter of the heater in Fig. 9–8.

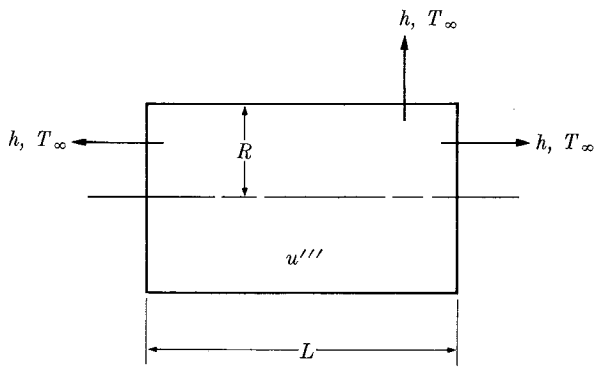


FIG. 9–7

For the system corresponding to a typical inner nodal point the first law of thermodynamics gives

$$0 = q_{i-1}2\pi\left(R_m - \frac{\Delta l}{2}\right)\Delta l + q_{j-1}2\pi R_m\,\Delta l + q_{i+1}2\pi\left(R_m + \frac{\Delta l}{2}\right)\Delta l + q_{j+1}2\pi R_m\,\Delta l + u'''2\pi R_m(\Delta l)^2. \tag{9–13}$$

* For further discussion of triangular and hexagonal networks see, for example, Reference 3, Chapter 10 and Reference 11, Chapter III.

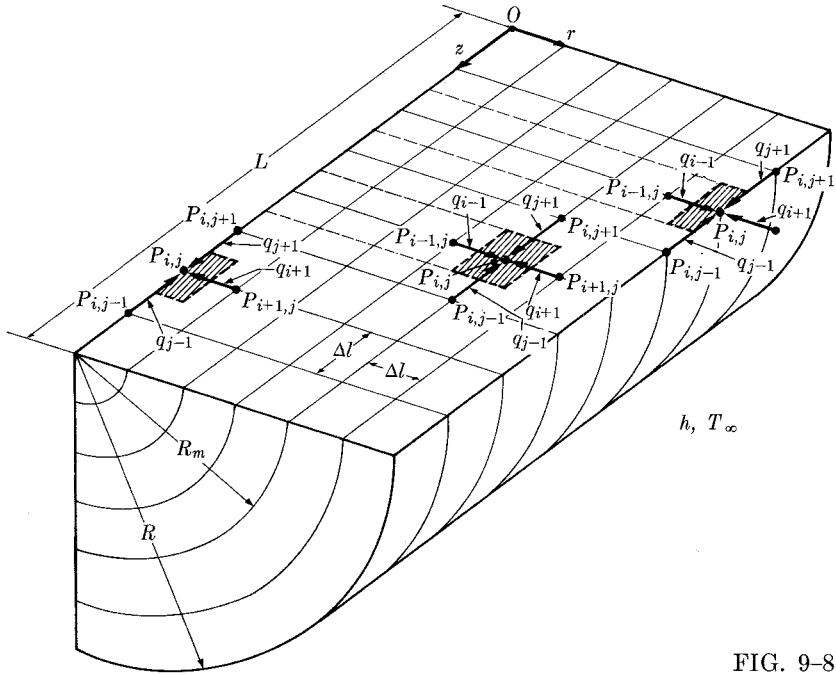


FIG. 9–8

We may relate the heat fluxes of Eq. (9–13) to the temperatures by Fourier's law as stated by Eq. (9–2), provided Eq. (9–2) is interpreted in terms of polar coordinates and a common difference length Δl is used; thus we obtain the difference equation

$$T_{i,j-1} + T_{i,j+1} + \left(1 - \frac{\Delta l}{2R_m}\right) T_{i-1,j} + \left(1 + \frac{\Delta l}{2R_m}\right) T_{i+1,j} - 4T_{i,j} + \frac{u'''(\Delta l)^2}{k} = 0. \qquad (9\text{–}14)$$

For the system corresponding to a typical centerline nodal point the first law of thermodynamics gives

$$0 = q_{j-1} \frac{\pi(\Delta l)^2}{4} + q_{i+1} \pi \, \Delta l \, \Delta l + q_{j+1} \frac{\pi(\Delta l)^2}{4} + u''' \frac{\pi(\Delta l)^2}{4} \Delta l. \qquad (9\text{–}15)$$

Expressing the heat fluxes of Eq. (9–15) by means of the appropriate form of Fourier's law, we have

$$T_{i+1,j} + \tfrac{1}{4}(T_{i,j-1} + T_{i,j+1}) - \tfrac{3}{2}T_{i,j} + \frac{u'''(\Delta l)^2}{4k} = 0. \qquad (9\text{–}16)$$

Similarly, for the typical boundary nodal point indicated in Fig. 9–8 we get

$$\left(1 - \frac{\Delta l}{2R}\right) T_{i-1,j} + \tfrac{1}{2}(T_{i,j-1} - T_{i,j+1}) - \left(2 + \frac{h\,\Delta l}{k} - \frac{\Delta l}{2R}\right) T_{i,j} + \frac{h\,\Delta l}{k} T_\infty + \frac{u'''(\Delta l)^2}{2k} = 0. \qquad (9\text{–}17)$$

(How many types of difference equation does the foregoing problem have?)

The difference equations given by Eqs. (9–14) and (9–16) may also be obtained by the mathematical approach, that is, by considering the term-by-term difference approximation of the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{u'''}{k} = 0. \qquad (9\text{–}18)$$

However, it should be noted that the nodal points on the axis $r = 0$ are exceptional; there the term $(1/r)(\partial T/\partial r)$ in Eq. (9–18) is indeterminate and must be replaced by its limit as $r \to 0$. It follows from L'Hôpital's rule that

$$\lim_{r\to 0} \frac{1}{r}\frac{\partial T}{\partial r} = \lim_{r\to 0} \frac{\partial T/\partial r}{r} \to \frac{\partial^2 T}{\partial r^2}.$$

Because of the inherent nature of cylindrical problems, the use of a logarithmic scale in the radial direction improves the difference formulation and hence the accuracy of the solution of these problems, but at the expense of increased labor. When the boundaries of a cylindrical problem are arcs of concentric cylinders, it is possible to bring these boundaries into a more convenient (cartesian) shape by conformal mapping.*

9–6. Irregular Boundaries

Let us reconsider Example 2–1 as discussed in Section 9–1, but assume that the boundaries of the electric heater are now irregular and therefore not suitable to simple coordinate systems (Fig. 9–9). In practice, such irregular geometry often occurs—for example, in turbine blades. This type of problem could be solved by approximating the boundaries in terms of, say, a cartesian coordinate system as shown in Fig. 9–9(b). However, the approximation thus introduced may not always be desirable. It is therefore of vital importance that we be able to apply the techniques of difference formulation to irregular geometry. This may, in fact, be done, although only at the expense of the introduction of modified difference equations. Figure 9–9(a) and its elaboration in Fig. 9–9(c) show the typical system suitable to irregular geometry. The first law of thermo-

* See, for example, Reference 7, p. 317.

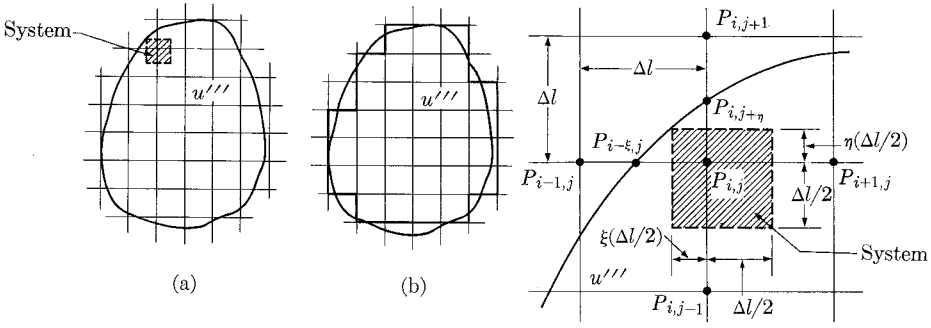


FIG. 9–9

dynamics applied to this system and rearranged by Fourier's law yields

$$\frac{2}{1+\xi}T_{i+1,j} + \frac{2}{1+\eta}T_{i,j-1} + \frac{2}{\xi(1+\xi)}T_{i-\xi,j} + \frac{2}{\eta(1+\eta)}T_{i,j+\eta} - 2\left(\frac{1}{\eta}+\frac{1}{\xi}\right)T_{i,j} + \frac{u'''(\Delta l)^2}{k} = 0. \qquad (9\text{–}19)$$

Equation (9–19) reduces, as of course it should, to Eq. (9–3) as $\xi \to 1$ and $\eta \to 1$. By a procedure analogous to that used in Section 9–3, we may also obtain Eq. (9–19) using the appropriate Taylor series expansion of Eq. (9–7). It is possible to show by this procedure that the terms neglected in Eq. (9–19) are of $O[(\Delta l)^3]$. Therefore, for a given size of network, there is more approximation introduced in a boundary difference equation when the boundary is irregular than when it is regular.

9–7. Solution of Steady Problems. Relaxation Method

In the preceding sections we have learned how to reduce the formulation of steady problems to a set of algebraic equations by means of the difference method. We know from mathematics, in principle at least, how to solve these equations by elimination or by determinants. However, when the number of equations increases, say to 10, 20, or even more, the labor required in the use of these procedures becomes excessive, and we are forced to employ other techniques, such as the use of *high-speed computers* and the *relaxation method.*

Computers may be classified as *analog* and *digital* computers. Analog computers include devices which replace the physical variables by electric potentials; their over-all accuracy is limited to the range 0.1–5%. In digital computation all numbers are quantized; there is no clear bound on the accuracy attainable. Despite the astonishingly rapid development of computers in recent years, common-language digital computers are not yet available. Therefore, there is still no place for digital computer solutions in a text of this nature. The analog solution of unsteady problems will be discussed in the next chapter.

Here we consider the solution of algebraic sets by the relaxation method only. However, the reader is urged to solve suitable problems of this chapter by means of available digital computers.

The basic rule of relaxation and some of its widely used modifications may best be explained by employing them to solve a simple pair of algebraic equations,

$$3x - y = 65, \qquad -x + 2y = 40. \tag{9-20}$$

We first rearrange these equations with all their terms on one side, then replace the zeros on the other side by the *residuals* R_1 and R_2 as follows:

$$R_1 = -3x + y + 65, \qquad R_2 = x - 2y + 40. \tag{9-21}$$

The basic rule of relaxation is to start with a pair of x and y, and while introducing orderly changes in the values of x and y, to reduce the currently larger residual to zero, or as nearly zero as desired. The starting values for x and y are, in general, arbitrary for algebraic equations, so they may conveniently be taken equal to zero. However, for problems of physical significance, a thorough understanding of the problem and past experience with similar problems may make it possible to select starting values different from zero, thus shortening the labor of solution to a large extent.

An *operation table*, which shows the effect on the residuals of a positive unit increment in the variables x and y, is convenient in the reduction of residuals. For the problem under consideration, the operation table is

	ΔR_1	ΔR_2
$\Delta x = 1$	-3	1
$\Delta y = 1$	1	-2

(9-22)

where Δ denotes an incremental change in a quantity. Note that the change Δx or Δy affects both residuals.

Starting with the value of zero for both x and y, we may systematically carry out the algebra in a *relaxation table* as shown below.

x	y	R_1	R_2
0	0	65	40
22		-1	62
	31	30	0
10		0	10
	5	5	0
2		-1	2
	1	0	0
34	37	0	0

(9-23)

Thus, following the basic rule of relaxation, first we change the value of x by $\Delta x = 22$ in order to reduce the larger residual (almost) to zero. This operation decreases x by -66 and increases y by 22, as may readily be seen from the operation table. The same operation reduces R_1 to -1 and increases R_2 to 62 in the relaxation table. Second, we increase the value of y by $\Delta y = 31$, which reduces R_2 to zero and increases R_1 to 30 in the relaxation table, and so on. Note the convergence of the process by observing the continual decrease in the sum of the residuals as the process continues. Since in the last line of the table the residuals are reduced to zero, the final values $x = 34$ and $y = 37$ actually are the exact solution of Eq. (9–20). This may easily be verified by inserting these values into one equation of Eq. (9–20).

In the preceding relaxation table, first we worked with integer numbers and obtained a solution correct to the nearest integers. If necessary, the calculations may be refined to the first decimal place by means of decimal increments, and so on.

One important advantage of the relaxation method is that if errors are made in the process of solving a problem, we do not need to go back and check each step of the solution procedure. The effect of errors may quickly be removed starting from the erroneous result. For example, in the pair of algebraic equations under consideration, suppose that the value 65 of R_1 in the relaxation table (9–23) is accidentally taken to be 55, and the table is completed accordingly, as shown on the left below.

x	y	R_1	R_2	
0	0	(55)	40	
18		1	58	Error
	29	30	0	
10		0	10	
	5	5	0	
2		−1	2	
	1	0	0	
30	35	10	0	

x	y	R_1	R_2
30	35	10	0
3		1	3
	2	3	−1
1		0	0
34	37	0	0

The error becomes evident from the fact that R_1 in the last line, obtained by inserting $x = 30$ and $y = 35$ into the first equation of Eq. (9–21), does not agree with R_1 in the preceding line, obtained by the relaxation procedure. We do not need to know the location of the error—we simply carry on the table until the residuals are actually reduced to zero, as shown on the right above. This time, inserting the values 34 and 37 into Eq. (9–21), we observe the agreement between residuals. No relaxation procedure may be assumed complete without this final check.

The basic rule of relaxation, the continual and systematic reduction of residuals to zero, may readily be applied to the solution of more complex prob-

lems. However, by three widely used modifications of this rule—*overrelaxation*, *block relaxation*, and *group relaxation*—and with a little attention, considerable time and effort can be saved. We next consider these modifications.

It is clear even before numerical computations that the use of the basic rule in the first step of relaxation reduces R_1 but increases R_2; in the second step R_2 is reduced but R_1 is increased, and so on. Now, instead of reducing R_1 in the first step, it is, of course, possible to deliberately make it negative such that the second step yields a smaller value for R_1 than that obtained by the basic rule. This procedure is known as *overrelaxation*. To what extent R_1 should be made negative is a matter of experience. However, assuming the convergence of residuals as the relaxation procedure continues, we may take an increment of about half as much as that which reduces R_1 to zero. Thus, instead of the increment $\Delta x = 22$ obtained by the basic rule, we deliberately overrelax x about 50% and take $\Delta x = 33$. The remainder of the relaxation table is completed by the basic rule, as shown below.

	x	y	R_1	R_2
	0	0	65	40
Overrelaxation	33		−34	73
		37	3	−1
	1		0	0
	34	37	0	0

A second and even more important modification of the basic rule is known as *block relaxation*. This procedure consists of using a unit block operation; that is, we apply increments to all unknowns simultaneously. For our illustrative example the unit block operation is $\Delta x = \Delta y = 1$; the corresponding change in residuals may be calculated directly from Eq. (9–21) or by adding together the changes produced by the basic unit operations in the operation table (9–22). Thus we have the operation table shown below.

		ΔR_1	ΔR_2
	$\Delta x = 1$	−3	1
	$\Delta y = 1$	1	−2
Unit block operation	$\Delta x = \Delta y = 1$	−2	−1

We now start a new relaxation table, reducing the initial value of both residuals simultaneously by a block relaxation. Note that the sum of the initial residuals is $105(= 65 + 40)$, and the unit block operation affects a change in that sum of $-3(= -2 - 1)$. First considering the block operation $\Delta x = \Delta y = 35(105/3)$, next completing the table by basic operations, we obtain the following table.

Block relaxation

x	y	R_1	R_2
0	0	65	40
35	35	−5	5
	2	−3	1
−1		0	0
34	37	0	0

The third modification of the basic rule is *group relaxation*, which consists of using different increments on some unknowns. In general, in a group relaxation our objective is to cause certain residuals to be of the same order but different in sign. These residuals may then be rapidly reduced by basic operations. The extension of our operation table (9–22) by a suitable group operation is as follows.

Group operation

	ΔR_1	ΔR_2
$\Delta x = 1$	−3	1
$\Delta y = 1$	1	−2
$\Delta x = 3, \Delta y = 4$	−5	−5

In terms of this group operation we begin a relaxation table with a group relaxation and obtain the table shown below.

Group relaxation

x	y	R_1	R_2
0	0	65	40
30	40	15	−10
5		0	−5
	−3	−3	1
−1		0	0
34	37	0	0

Because of the triviality of our illustrative example, we had to employ overrelaxation, block relaxation, and group relaxation procedures at the beginning of our relaxation tables. In actual cases these procedures should be adopted at whatever point they are appropriate.

Here we terminate our study of the relaxation method. However, the reader is urged to consider, within the spirit of the method, the possibility of further modifications of the basic rule of relaxation in conjunction with special situations. Now, starting with a trivial example, we illustrate the use of the relaxation method by applying it to the solution of three examples of physical significance.

Example 9–1. Consider a flat plate of thickness $2L = 4$ in. Assume that internal energy is generated in this plate at the rate of

$$u''' = 14{,}400 \text{ Btu/ft}^3\text{·hr}.$$

The thermal conductivity of the plate is $k = 10$ Btu/ft·hr·°F. The ambient temperatures are 50°F and 100°F. The heat transfer coefficients are large. We wish to find the temperature distribution in the plate.

Let us assume a somewhat crude network, and divide the plate into four equal parts of thickness Δx. The temperatures of the nodal points are denoted by T_0, T_1, T_2, T_3, and T_4. Since the surface temperatures are given, the only temperatures to be determined are T_1, T_2, and T_3.

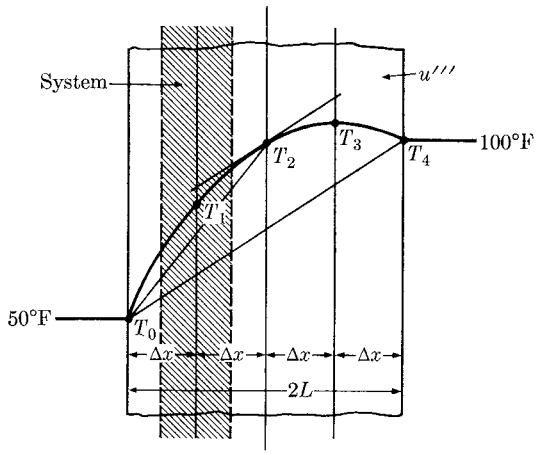


FIG. 9–10

We apply the first law of thermodynamics to the typical system shown in Fig. 9–10; the result related to the temperatures by Fourier's law gives

$$0 = T_0 + T_2 - 2T_1 + u'''(\Delta x)^2/k.$$

The difference equations corresponding to nodal points 2 and 3 are the same, provided the temperature subscripts are changed accordingly. Noting that

$$\frac{u'''(\Delta x)^2}{k} = \frac{14{,}400(1/12)^2}{10} = 10°\text{F}, \tag{9–24}$$

and introducing the residuals R_1, R_2, and R_3, we have

$$\begin{aligned} R_1 &= T_2 - 2T_1 + 60, \\ R_2 &= T_1 + T_3 - 2T_2 + 10, \\ R_3 &= T_2 - 2T_3 + 110. \end{aligned} \tag{9–25}$$

We then have the following operation table.

		ΔR_1	ΔR_2	ΔR_3
	$\Delta T_1 = 1$	−2	1	0
	$\Delta T_2 = 1$	1	−2	1
	$\Delta T_3 = 1$	0	1	−2
Block	$\Delta T_1 = \Delta T_2 = \Delta T_3 = 1$	−1	0	−1

In the absence of energy generation the temperature distribution would be linear, and we would have $T_1 = 62.5°F$, $T_2 = 75°F$, and $T_3 = 87.5°F$. The energy generation should increase these temperatures by about 10°F according to Eq. (9–24). However, if we add this amount to T_2, then T_1 and T_3 must be augmented by an amount between 5°F and 10°F.* Assuming, for example, 7.5°F as the temperature to be added to these temperatures, we may start our relaxation table with

T_1	R_1	T_2	R_2	T_3	R_3
70	5	85	5	95	5

where the residuals have been evaluated from Eq. (9–25). The equality of these residuals suggests the search for a group operation affecting the residuals equally. In fact, after a few trials, we find the appropriate extension of the operation table.

		ΔR_1	ΔR_2	ΔR_3
Group I	$\Delta T_1 = \Delta T_3 = 1, \quad \Delta T_2 = 2$	0	−2	0
Group II = Group I + 2 Block	$\Delta T_1 = \Delta T_3 = 3, \quad \Delta T_2 = 4$	−2	−2	−2

Now the relaxation table may be easily completed as shown below.

T_1	R_1	T_2	R_2	T_3	R_3
70	5	85	5	95	5
7.5	0	10	0	7.5	0
77.5	0	95	0	102.5	0

* See the temperature distribution sketched in Fig. 9–10, and note the straight line connecting T_0 to T_2.

Despite the crude network used, the foregoing temperatures correspond to the *exact* values of these temperatures (why?).

Example 9–2. Two surfaces of a square rod are kept at 100°F, while the other two surfaces are kept at 0°F. We wish to find the temperature distribution in the rod.

Let us assume the network shown in Fig. 9–11. Clearly, a finer network would be necessary to represent the temperature distribution; here, however, our purpose is to demonstrate the features of the relaxation method rather than to obtain an accurate solution. Because of the thermal symmetry of the problem, we need consider only the temperature of the four locations indicated in the figure. The corresponding difference equations, modified by the residuals, are

$$\begin{aligned} R_1 &= 2T_2 + 2T_3 - 4T_1, \\ R_2 &= T_1 + 2T_4 + 100 - 4T_2, \\ R_3 &= T_1 + 2T_4 - 4T_3, \\ R_4 &= T_2 + T_3 + 100 - 4T_4. \end{aligned}$$

Thus we have the operation table shown below.

	ΔR_1	ΔR_2	ΔR_3	ΔR_4
$\Delta T_1 = 1$	−4	1	1	0
$\Delta T_2 = 1$	2	−4	0	1
$\Delta T_3 = 1$	2	0	−4	1
$\Delta T_4 = 1$	0	2	2	−4

Now we may start the relaxation table by guessing the temperature of locations 1, 2, 3, and 4. The exact temperature of location 1 is $T_1 = 50$°F (why?). Then a linear interpolation gives $T_2 = 75$°F and $T_3 = 25$°F; the corresponding residuals are as follows.

T_1	R_1	T_2	R_2	T_3	R_3	T_4	R_4
50	0	75	−50	25	50	50	0

The values of these residuals suggest the use of a group operation which inversely and equally affects R_2 and R_3 without influencing R_1 and R_4. Inspection of the part of the operation table already established reveals the desired grouping as shown below.

		ΔR_1	ΔR_2	ΔR_3	ΔR_4
Group	$\Delta T_2 = -1, \quad \Delta T_3 = 1$	0	4	−4	0

The appropriate difference equations, modified by the residuals, are

$$R_0 = 2T_1 + 100 - 4T_0,$$
$$R_1 = T_0 + T_2 + 100 - 4T_1,$$
$$R_2 = T_1 + T_3 + 100 - 4T_2,$$
$$R_3 = 2T_2 - 4T_3.$$

Then we have the following operation table.

	ΔR_0	ΔR_1	ΔR_2	ΔR_3
$\Delta T_0 = 1$	−4	1	0	0
$\Delta T_1 = 1$	2	−4	1	0
$\Delta T_2 = 1$	0	1	−4	2
$\Delta T_3 = 1$	0	0	1	−4

Next we start guessing the temperature of the nodal points. Because of the thickness of the chimney, the temperature is approximately one-dimensional in the neighborhood of the nodal point 0. Thus we have $T_0 = 50°\text{F}$ and, by assuming a parabolic distribution between the nodal points 0 and 4, $T_1 = 47°\text{F}$, $T_2 = 38°\text{F}$, and $T_3 = 22°\text{F}$. The complete relaxation table is shown below.

	T_0	R_0	T_1	R_1	T_2	R_2	T_3	R_3	Steps
	50	−6	47	0	38	17	22	−12	Start
				4	4	1		−4	1
Group I	−1	0	1	0	1	−3	−1	2	2
				−1	−1	1		0	3
		−0.6	−0.3	0.2		0.7			4
				0.4	0.2	−0.1		0.4	5
	−0.2	0.2		0.2					6
						0	0.1	0	7
Group II	0.1	0	0.1	−0.1		0.1			8
	48.9	0	47.8	−0.1	42.2	0.1	21.1	0	

Note that we start with the basic rule. Then the resulting residuals suggest the following group operation.

		ΔR_1	ΔR_2	ΔR_3	ΔR_4
Group I	$\Delta T_0 = \Delta T_3 = -1, \quad \Delta T_1 = \Delta T_2 = 1$	6	−4	−4	6

After the fourth step calculations are refined to the first decimal place; again

we employ the basic rule except in the eighth step, which implies the group operation shown below.

		ΔR_1	ΔR_2	ΔR_3	ΔR_4
Group II	$\Delta T_0 = \Delta T_1 = 1$	-2	-3	1	0

Inspection of the residuals of the third and eighth steps reveals that these differ by a decimal point only. This property allows us to carry the solution of difference equations to an unlimited number of decimal places. For example, the results to be obtained by carrying the calculations to the next decimal place would give $T_0 = 48.89$, $T_1 = 47.78$, $T_2 = 42.22$, and $T_3 = 21.11$, and the residuals $R_0 = 0$, $R_1 = -0.01$, $R_2 = 0.01$, and $R_3 = 0$. However, it should be remarked that the foregoing improvement in accuracy has no effect on the approximation involved in the difference formulation of the problem.

9–8. Difference Formulation of Unsteady Problems. Stability

As with steady problems, we introduce the difference formulation of unsteady problems in terms of an illustrative example taken from cartesian geometry. Thus we consider a flat electric heater plate of negligible thickness between two plates of thickness L, density ρ, specific heat c, and thermal conductivity k (Fig. 9–13). The energy generated in the heater is transferred through the plates to the ambient. The heat transfer coefficient is large, and the temperatures are measured relative to the ambient temperature.* Assume now that the heater is turned off. We wish to find the difference formulation of the subsequent unsteady problem.

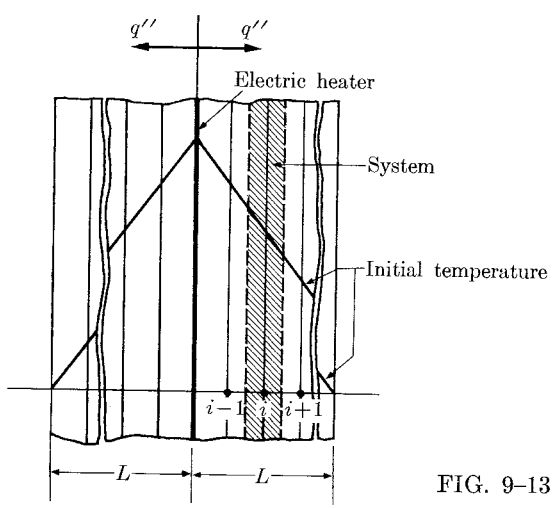


FIG. 9–13

* The initial steady problem is a special case of that associated with Fig. 2–15.

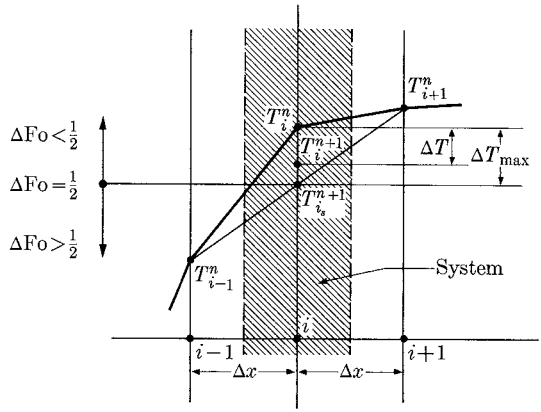


FIG. 9–14

Applying the first law of thermodynamics to the system shown and elaborated in Figs. 9–13 and 9–14, respectively, and relating the result to the temperatures by means of Fourier's law, we obtain

$$\rho c(\Delta x)\,\frac{T_i^{n+1} - T_i^n}{\Delta t} = k\,\frac{T_{i-1}^n - T_i^n}{\Delta x} + k\,\frac{T_{i+1}^n - T_i^n}{\Delta x}, \qquad (9\text{–}26)$$

and subsequently

$$T_i^{n+1} - T_i^n = \Delta \mathrm{Fo}\,(T_{i+1}^n - 2T_i^n + T_{i-1}^n), \qquad (9\text{–}27)$$

where, as before, the subscripts denote locations; the newly introduced superscripts denote time intervals, and $\Delta \mathrm{Fo} = a(\Delta t)/(\Delta x)^2$ is Fourier's modulus based on the difference values of time and space.

If we know the temperatures T_{i+1}^n, T_i^n, and T_{i-1}^n, as well as $\Delta \mathrm{Fo}$, the temperature change at location i from T_i^n to T_i^{n+1} during the time interval

$$(n + 1)\,\Delta t - n\,\Delta t$$

may be found from Eq. (9–27). Clearly, selecting some values for the intervals Δt and Δx, and getting the value of a (for the plate material) from a table of properties, we can evaluate $\Delta \mathrm{Fo}$. Postponing comment on the solution of unsteady problems to Section 9–9, let us assume for the time being that we have carried out the calculations for two slightly different values of $\Delta \mathrm{Fo}$, say $\frac{5}{11}$ and $\frac{5}{9}$. This corresponds to slightly different values for the time intervals if the plate is divided into the same equal subintervals, say 10, for both cases. The results are compared with the exact solution in Figs. 9–15 and 9–16. Inspection of these figures reveals that although the results corresponding to the value $\Delta \mathrm{Fo} = \frac{5}{11}$ are quite satisfactory, those corresponding to $\Delta \mathrm{Fo} = \frac{5}{9}$ are influenced by some sort of accumulated and amplified error. The phenomenon occurring in the latter case is called *instability*. The error thus introduced has nothing to do with the truncation error discussed in Section 9–2 or the *round-off*

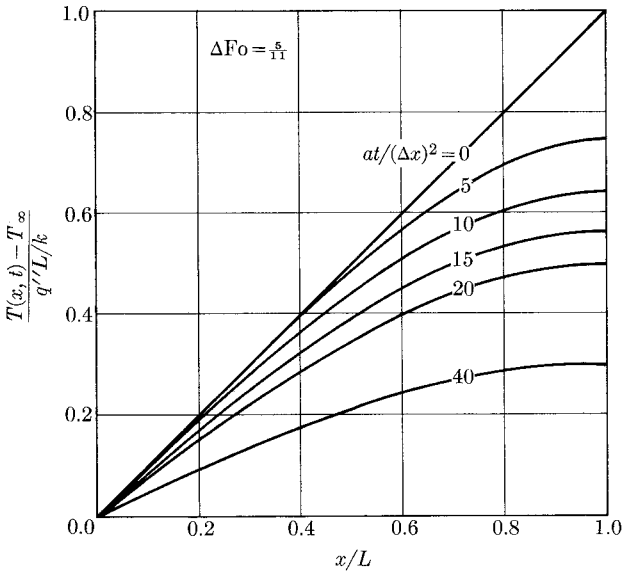


FIG. 9–15

error resulting from the use of rounded numbers, both of which are negligibly small in the present example; it is, rather, a property of difference equations, and it increases with successively smaller values of Δx unless Δt is also reduced properly. We therefore learn the important fact that *the difference formulation of unsteady problems is never complete without the statement of the stability criteria for the difference equations.* Formal but somewhat lengthy mathematical proofs on stability may be found in the literature.* Here we suggest a rather simple physical argument based on the approximation involved in the problems of field theory: *the spacewise variation of volume terms is negligible as compared with their timewise variation; the timewise variation of surface terms is negligible as compared with their spacewise variation.*† Actually, we have always used this approximation in the formulation of our problems without emphasizing it. For example, in the development of Eq. (9–26) we neglected the spacewise variation in the internal energy, which is a volume term, and the timewise variation in the heat fluxes by conduction, which are surface terms. Therefore, in the time interval $(n + 1)\,\Delta t - n\,\Delta t$, the temperature T_i^n changes to T_i^{n+1} while the temperatures T_{i-1}^n and T_{i+1}^n remain constant in order of magnitude. The limit of T_i^{n+1} is its steady value T_{is}^{n+1}, which may be obtained by assuming that T_i^{n+1} reaches this value in the interval $(n + 1)\,\Delta t - n\,\Delta t$. Since the steady one-dimensional temperature distribution in a plate of constant thermal conductivity is a straight line, T_{is}^{n+1} is at the intersection of the isothermal plane of location i with a straight line connecting T_{i-1}^n and T_{i+1}^n. Correspond-

* See, for example, Reference 4.
† Recall Problem 5–7.

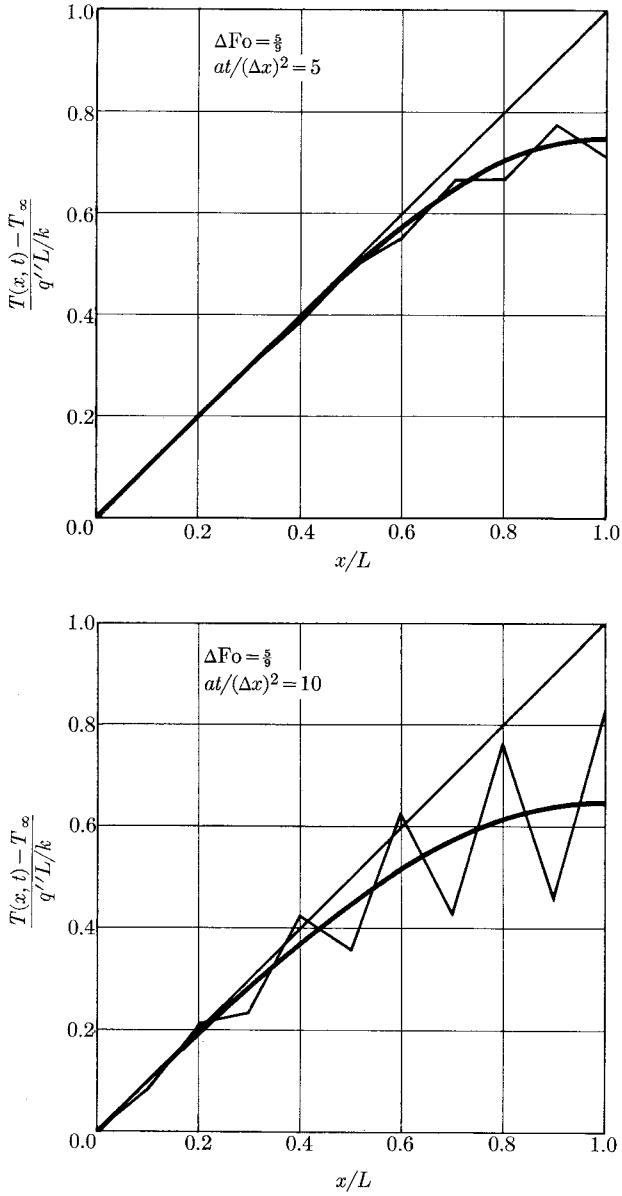


FIG. 9–16

ingly, the limiting steady form of Eq. (9–26) or Eq. (9–27) is

$$0 = T^n_{i+1} - 2T^{n+1}_{i_s} + T^n_{i-1}. \tag{9–28}$$

Now, subtraction of Eq. (9–28) from (Eq. 9–27) and subsequent rearrangement

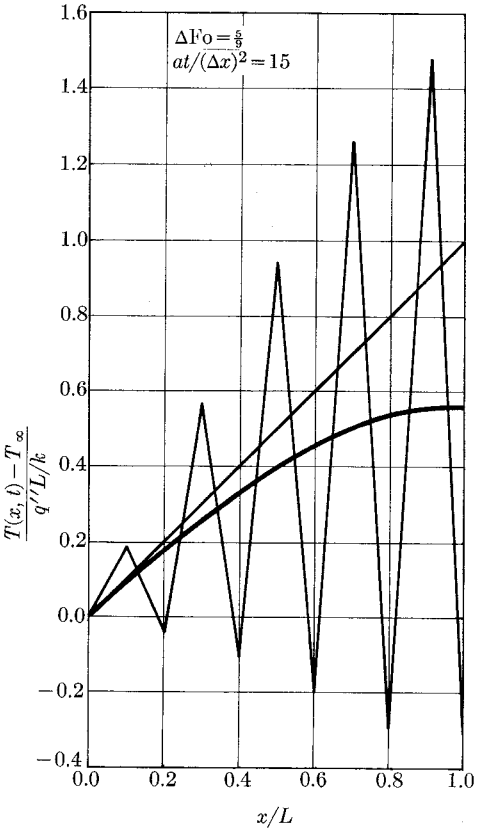


FIG. 9–16 (*cont.*)

in terms of the nomenclature of Fig. 9–14 gives

$$\Delta T = 2(\Delta \text{Fo})\, \Delta T_{\max}.$$

Furthermore, noting from the same figure that $\Delta T \leq \Delta T_{\max}$, we find that the stability criterion of Eq. (9–27) is

$$\Delta \text{Fo} \leq \tfrac{1}{2}. \tag{9–29}$$

Thus the reason behind the selection of the values $\frac{5}{11}$ and $\frac{5}{9}$ for ΔFo in Figs. 9–15 and 9–16 becomes clear. It is remarkable that the case $\Delta \text{Fo} > \frac{1}{2}$, which causes instability in the difference equations, also violates the physics of the problem by giving a temperature that exceeds its steady value. For the limiting case $\Delta\,\text{Fo} = \frac{1}{2}$, Eq. (9–27) reduces to Eq. (9–28) or, equivalently, to

$$T_{i_s}^{n+1} = \frac{T_{i+1}^{n} + T_{i-1}^{n}}{2}. \tag{9–30}$$

Equation (9–30) readily admits a graphical interpretation as follows. The temperature change $T_{i_s}^{n+1}$ at location i in the time interval $(n + 1)\,\Delta t - n\,\Delta t$

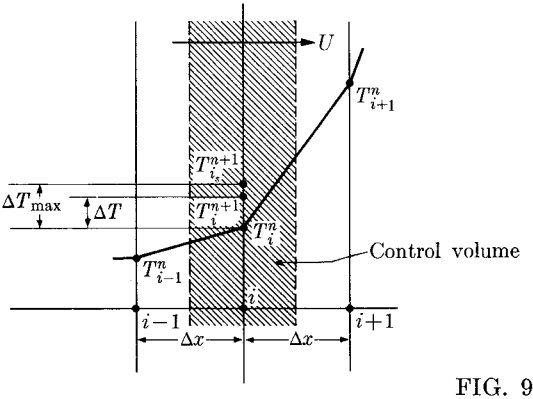


FIG. 9–17

is the arithmetic mean of the neighboring temperatures T^n_{i+1} and T^n_{i-1} taken from the preceding interval; thus the intersection of the straight line connecting T^n_{i+1} to T^n_{i-1} with the isothermal plane of location i gives $T^{n+1}_{i_s}$ (Fig. 9–14). This is the basis of the *Binder-Schmidt graphical method,* which will be discussed in Section 9–9.

We continue our discussion by considering a moving one-dimensional continuum (Fig. 9–17).* The first law of thermodynamics and Fourier's law successively applied to the control volume of Fig. 9–17 give

$$\rho c(\Delta x)\frac{T^{n+1}_i - T^n_i}{\Delta t} = \rho cU(T^n_{i-1} - T^n_i) + k\frac{T^n_{i-1} - T^n_i}{\Delta x} + k\frac{T^n_{i+1} - T^n_i}{\Delta x},$$

which may be rearranged in the form

$$T^{n+1}_i - T^n_i = \Delta\text{Fo}\,[T^n_{i+1} - (2 + \Delta\text{Pé})\,T^n_i + (1 + \Delta\text{Pé})T^n_{i-1}], \qquad (9\text{–}31)$$

where $\Delta\text{Pé} = U(\Delta x)/a$ is the Péclèt modulus based on the difference value of the space variable. Proceeding as in the stationary case, we find that the limiting steady value of Eq. (9–31) based on the temperatures T^n_{i+1}, $T^{n+1}_{i_s}$, and T^n_{i-1} is

$$0 = \Delta\text{Fo}\,[T^n_{i+1} - (2 + \Delta\text{Pé})\,T^{n+1}_{i_s} + (1 + \Delta\text{Pé})\,T^n_{i-1}]. \qquad (9\text{–}32)$$

Subtracting Eq. (9–32) from Eq. (9–31), and noting that $\Delta T \leq \Delta T_{max}$, we obtain the stability criterion for Eq. (9–31) in the form

$$\Delta\text{Fo} \leq 1/(2 + \Delta\text{Pé}). \qquad (9\text{–}33)$$

Strictly speaking, the graphical interpretation of Eq. (9–32), which actually constitutes the graphical solution of the equation, belongs in Section 9–9.

* Recall the irrelevance of the temperature distribution with regard to the formulation.

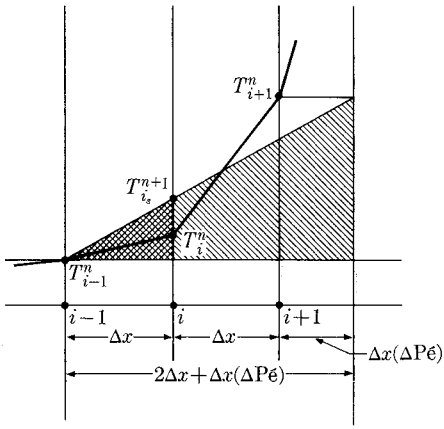


FIG. 9–18

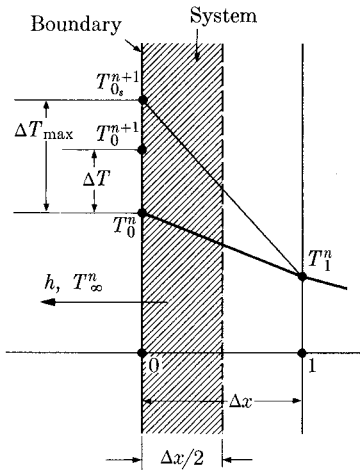


FIG. 9–19

However, because of its simplicity this interpretation is given here. Rearranging Eq. (9–32), and dividing the result by Δx, we have

$$\frac{T^{n+1}_{i_s} - T^n_{i-1}}{\Delta x} = \frac{T^n_{i+1} - T^n_{i-1}}{2\,\Delta x + \Delta x(\Delta \text{Pé})}. \tag{9–34}$$

The similarity of the triangles shown in Fig. 9–18 readily yields Eq. (9–34). Note that the temperature $T^{n+1}_{i_s}$ can no longer be obtained by connecting the neighboring temperatures with a straight line, because of the terms in Eq. (9–32) related to enthalpy flow. (What happens to Fig. 9–18 as $U \to 0$?)

For two particular problems, so far we have established difference equations applicable to a typical inner nodal point. Now, we turn to boundary nodal points and assume, for example, heat transfer by convection through the boundary of a stationary one-dimensional problem. The first law of thermodynamics and Fourier's law successively applied to the system shown in Fig. 9–19 yield

$$\rho c\left(\frac{\Delta x}{2}\right)\frac{T^{n+1}_0 - T^n_0}{\Delta t} = h(T^n_\infty - T^n_0) + k\,\frac{T^n_1 - T^n_0}{\Delta x}, \tag{9–35}$$

where T^n_∞ denotes the ambient temperature in the time interval n. Rearranging Eq. (9–35), we obtain

$$T^{n+1}_0 - T^n_0 = 2\Delta\text{Fo}\,[T^n_1 - (1 + \Delta\text{Bi})\,T^n_0 + (\Delta\text{Bi})\,T^n_\infty], \tag{9–36}$$

where $\Delta\text{Bi} = h\,\Delta x/k$ denotes the Biot modulus based on the difference value of the space variable. For the limiting steady case, we also have

$$0 = T^n_1 - (1 + \Delta\text{Bi})\,T^{n+1}_{0_s} + (\Delta\text{Bi})\,T^n_\infty. \tag{9–37}$$

In the usual manner, subtracting Eq. (9–37) from Eq. (9–36) and noting again that $\Delta T \leq \Delta T_{\max}$, we obtain the stability criterion for Eq. (9–36) as

$$\Delta \mathrm{Fo} \leq 1/2(1 + \Delta \mathrm{Bi}). \tag{9–38}$$

Similarly, Eq. (9–37) readily accepts a geometric interpretation when rearranged in the form

$$\frac{T_\infty^n - T_{0_s}^{n+1}}{\Delta x/\Delta \mathrm{Bi}} = \frac{T_{0_s}^{n+1} - T_1^n}{\Delta x}. \tag{9–39}$$

Now, if we extend the boundary of the problem by the distance $\Delta x/\Delta \mathrm{Bi}$ and assume that the extended boundary has the temperature T_∞^n in the time interval n, the intersection of the straight line connecting T_∞^n to T_1^n with the original boundary gives $T_{0_s}^{n+1}$ (Fig. 9–20). Clearly, the similarity of the triangles indicated in Fig. 9–20 satisfies Eq. (9–39).

In preceding chapters we have indicated the difficulty of solving multidomain problems. Only in Chapter 7 were we able to consider even two-domain problems.* The difference formulation and its solution by numerical or graphical means, however, present no difficulty and hence are indispensable for these problems. Consider, for example, a composite wall made of two parallel flat plates. The difference equation given by Eq. (9–27) is directly applicable to the inner nodal points of both plates provided $\Delta \mathrm{Fo} \leq \frac{1}{2}$. Excluding for now the graphical case corresponding to $\Delta \mathrm{Fo} = \frac{1}{2}$, let us assume the values

$$\Delta \mathrm{Fo}_1 = m_1 \quad \text{and} \quad \Delta \mathrm{Fo}_2 = m_2, \tag{9–40}$$

where $m_1, m_2 < \frac{1}{2}$. Since the time interval to be used must be the same for both plates, we obtain from Eq. (9–40) the relation between the space intervals of the plates in the form

$$\frac{\Delta x_1}{\Delta x_2} = \left(\frac{a_1}{a_2}\,\frac{m_2}{m_1}\right)^{1/2}. \tag{9–41}$$

Therefore, we need to select the interval size of only one plate; Eq. (9–41) gives the interval size of the other.

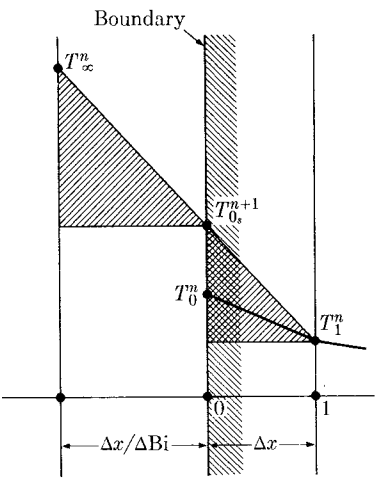


FIG. 9–20

Unlike the difference equation for the inner nodal points, the difference equation for the interface is not available from preceding sections. However, the derivation of this equation is straightforward. The result in terms of the

* See Example 7–15.

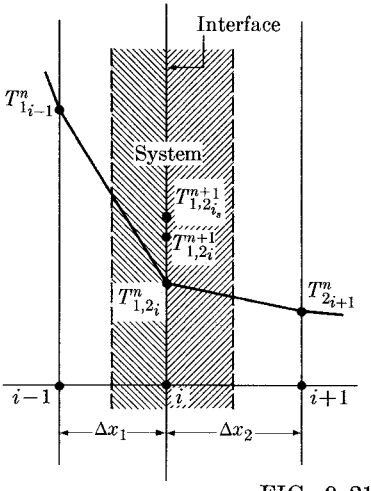


FIG. 9–21

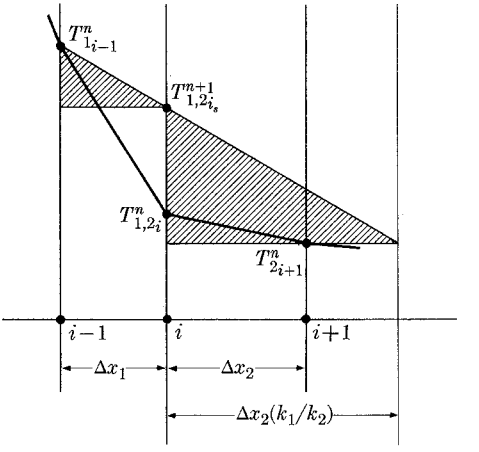


FIG. 9–22

system shown in Fig. 9–21 is

$$T^{n+1}_{1,2_i} - T^n_{1,2_i} = \frac{(k_2/\Delta x_2)T^n_{2_{i+1}} - (k_1/\Delta x_1 + k_2/\Delta x_2)T^n_{1,2_i} + (k_1/\Delta x_1)T^{n_i}_{1_{-1}}}{(\rho_1 c_1\,\Delta x_1 + \rho_2 c_2\,\Delta x_2)/2\,\Delta t}; \tag{9–42}$$

the steady solution is

$$0 = (k_2/\Delta x_2)T^n_{2_{i+1}} - (k_1/\Delta x_1 + k_2/\Delta x_2)T^{n+1}_{1,2_{i,s}} + (k_1/\Delta x_1)T^n_{1_{i-1}}, \tag{9–43}$$

and the stability criterion for Eq. (9–42) is

$$\left(\frac{k_1/\Delta x_1 + k_2/\Delta x_2}{\rho_1 c_1\,\Delta x_1 + \rho_2 c_2\,\Delta x_2}\right)\Delta t \leq \frac{1}{2}. \tag{9–44}$$

[Note the special case of Eq. (9–44) for $k_1 = k_2$ and $\rho_1 c_1 = \rho_2 c_2$.] The geometric interpretation of Eq. (9–43) may be obtained from its rearrangement in the form

$$\frac{T^n_{1_{i-1}} - T^{n+1}_{1,2_{i,s}}}{\Delta x_1} = \frac{T^{n+1}_{1,2_{i,s}} - T^{n.}_{2_{i+1}}}{\Delta x_2(k_1/k_2)}, \tag{9–45}$$

which is readily satisfied by the similarity of the triangles shown in Fig. 9–22. However, it is easier to consider the graphical procedure in terms of the intervals $\Delta\xi_1 = \Delta x_1/k_1$ and $\Delta\xi_2 = \Delta x_2/k_2$, thereby eliminating the use of Fig. 9–22. In this case, introducing the variable transformation $\xi = x/k$, we find the differential formulation of the problem to be

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c k}\frac{\partial^2 T}{\partial \xi^2}. \tag{9–46}$$

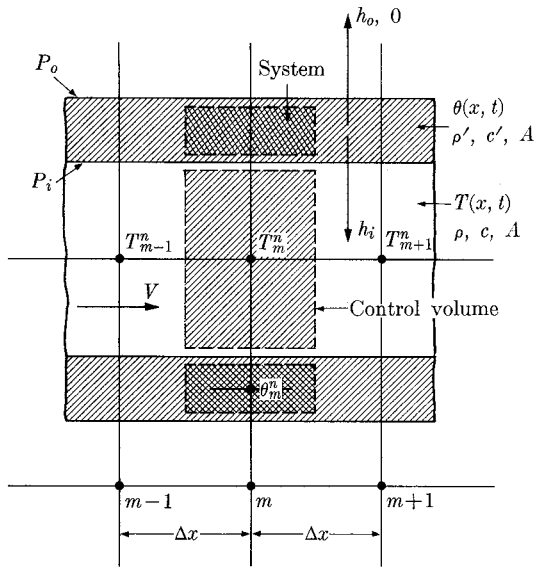


FIG. 9–23

Thus $1/\rho ck$ replaces a, and the stability criterion for the difference equation corresponding to Eq. (9–46) is

$$\frac{\Delta t}{\rho ck(\Delta \xi)^2} \leq \frac{1}{2}. \tag{9–47}$$

Considering the equality in Eq. (9–47) for both plates, and equating the time intervals, we obtain as the relation between the new space intervals,

$$\frac{\Delta \xi_1}{\Delta \xi_2} = \left(\frac{\rho_2 c_2 k_2}{\rho_1 c_1 k_1}\right)^{1/2}. \tag{9–48}$$

The extension of the difference formulation to unsteady problems of other geometries and the geometric interpretation of its limiting steady case require no additional knowledge. Therefore any elaboration on these points seems unnecessary.

Next we illustrate by an example how physical reasoning can readily be used to establish the stability criteria of complicated problems.

Example 9–4. Let us find the difference equations and their stability criteria for the typical inner nodal point associated with Example 6–3.

The first law of thermodynamics and the definition of the heat transfer coefficient applied successively to the control volume for the fluid and the system for the tube walls shown in Fig. 9–23 give for the fluid

$$\rho cA(\Delta x)\frac{T^{n+1}_m - T^n_m}{\Delta t} = \rho cAV(T^n_{m-1} - T^n_m) + h_i P_i(\Delta x)(\theta^n_m - T^n_m),$$

and for the tube walls

$$\rho' c' A'(\Delta x) \frac{\theta_m^{n+1} - \theta_m^n}{\Delta t} = h_i P_i(\Delta x)(T_m^n - \theta_m^n) + h_o P_o(\Delta x)(0 - \theta_m^n).$$

The rearrangement of these equations yields

$$T_m^{n+1} - T_m^n = \frac{V(\Delta t)}{\Delta x} T_{m-1}^n - \left(\frac{V}{\Delta x} + \frac{h_i P_i}{\rho c A}\right)(\Delta t) T_m^n + \frac{h_i P_i(\Delta t)}{\rho c A} \theta_m^n, \tag{9–49}$$

$$\theta_m^{n+1} - \theta_m^n = \frac{h_i P_i(\Delta t)}{\rho' c' A'} T_m^n - \left(\frac{h_i P_i + h_o P_o}{\rho' c' A'}\right)(\Delta t) \theta_m^n. \tag{9–50}$$

The corresponding steady equations are

$$0 = \frac{V(\Delta t)}{\Delta x} T_{m-1}^n - \left(\frac{V}{\Delta x} + \frac{h_i P_i}{\rho c A}\right)(\Delta t) T_{m_s}^{n+1} + \frac{h_i P_i(\Delta t)}{\rho c A} \theta_m^n, \tag{9–51}$$

$$0 = \frac{h_i P_i(\Delta t)}{\rho' c' A'} T_m^n - \left(\frac{h_i P_i + h_o P_o}{\rho' c' A'}\right)(\Delta t) \theta_{m_s}^{n+1}. \tag{9–52}$$

Subtracting Eq. (9–51) from Eq. (9–49), and Eq. (9–52) from Eq. (9–50), we obtain

$$T_m^{n+1} - T_m^n = \left(\frac{V}{\Delta x} + \frac{h_i P_i}{\rho c A}\right)(\Delta t)(T_{m_s}^{n+1} - T_m^n), \tag{9–53}$$

$$\theta_m^{n+1} - \theta_m^n = \left(\frac{h_i P_i + h_o P_o}{\rho' c' A'}\right)(\Delta t)(\theta_{m_s}^{n+1} - \theta_m^n). \tag{9–54}$$

Thus the stability criterion for Eq. (9–49) is

$$\left(\frac{V}{\Delta x} + \frac{h_i P_i}{\rho c A}\right) \Delta t \leq 1, \tag{9–55}$$

and that for Eq. (9–50) is

$$\left(\frac{h_i P_i + h_o P_o}{\rho' c' A'}\right) \Delta t \leq 1. \tag{9–56}$$

Needless to say, the common time interval to be used for the fluid and tube-wall difference equations must satisfy both Eq. (9–55) and Eq. (9–56).

So far, we have considered the difference formulation and its criteria of stability for a number of unsteady problems. The use of the formulations thus obtained becomes somewhat cumbersome when, in the interests of accuracy, a rather small Δx is selected; the Δt allowed by the stability criterion is then so small that an enormous amount of calculations may be required. We now

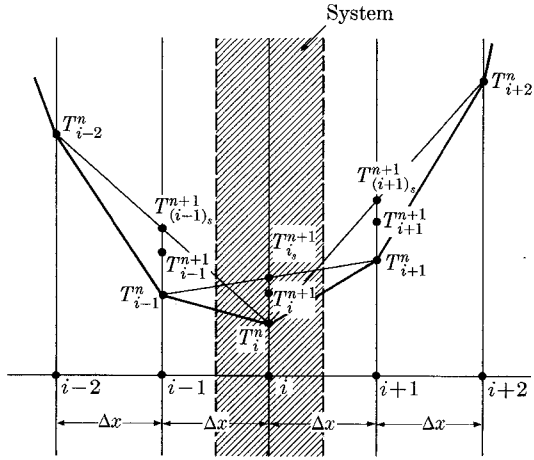


FIG. 9–24

intend to eliminate this difficulty by giving different forms to difference equations. Let us take the case of one-dimensional conduction in stationary problems for which we obtained the difference equation given by Eq. (9–27). Consider the system shown in Fig. 9–24 and formulate the problem this time in terms of *backward* rather than *forward* differences in time; that is, decrease the time from $(n + 1)\,\Delta t$ to $n\,\Delta t$ rather than following the usual procedure of increasing it from $n\,\Delta t$ to $(n + 1)\,\Delta t$. Thus we obtain

$$T_i^{n+1} - T_i^n = \Delta\text{Fo}\,(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}). \tag{9–57}$$

Subtracting from Eq. (9–57) its initial steady value given by Eq. (9–28) yields

$$T_i^{n+1} - T_i^n = \Delta\text{Fo}\,[(T_{i+1}^{n+1} - T_{i+1}^n) + 2(T_{i_s}^{n+1} - T_i^{n+1}) + (T_{i-1}^{n+1} - T_{i-1}^n)]. \tag{9–58}$$

Since all differences appearing in Eq. (9–58) are positive according to Fig. 9–24,* this equation is satisfied under all circumstances. Therefore, it is *unconditionally* stable. We obtain this stability, however, at the cost of a new algebraic complexity. Recall Eq. (9–27), in which all temperatures except T_i^{n+1} are known, and the latter is obtained by solution of the equation. By contrast, in Eq. (9–57) only T_i^n is known, and the application of this equation to the nodal points yields a set of simultaneous algebraic equations which must then be solved by the relaxation method discussed in Section 9–7 or by computer. Equations (9–27) and (9–57) are referred to as *explicit* and *implicit* difference equations, respectively. Although there exists only one explicit equation

* The only other possibility is a downward concave temperature distribution for which all differences are negative, and Eq. (9–58) is again unconditionally satisfied.

corresponding to each nodal point of a problem, the implicit equation may be written in a number of forms. For example, we may express the right-hand side of Eq. (9–57) in the form

$$T_i^{n+1} - T_i^n = \Delta\text{Fo}\,[\eta(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}) + (1 - \eta)(T_{i+1}^n - 2T_i^n + T_{i-1}^n)], \qquad (9\text{–}59)$$

where $0 \leq \eta \leq 1$. The reader is referred to Reference 4* for further discussion on this point.

The extension of the difference formulation to multidimensional problems is trivial, and will not be considered here.

9–9. Solution of Unsteady Problems. Step-by-Step Numerical Solution. Binder-Schmidt Graphical Method

Because of the inherent nature of the difference formulation of unsteady problems, we could not avoid discussing the numerical and graphical solutions of these problems in the process of their formulation. Here, after some recapitulation, we shall indicate a number of salient points of these solutions.

Unsteady problems are initial-value problems. The solution of their explicit difference formulation is trivial; it amounts to the evaluation of the temperature change at each location and in each time interval in terms of the temperature of the same location and the neighboring temperatures taken from the preceding time interval; the procedure is carried out numerically and step by step to following intervals. The solution of the implicit difference formulation reduces at each location and in each time interval to an algebraic equation involving the temperature of the same location and the neighboring temperatures expressed in terms of the following time interval; thus we obtain a set of algebraic equations to be solved by iteration, either by the relaxation method or by computer.

It is important to note that although steady problems are in general boundary-value problems, those having motion and negligible conduction in a given direction are conceptually initial-value problems. There have been some attempts to convert initial-value problems to boundary-value problems by means of a transformation which doubles the order of each derivative involved in the former. However, these methods have remained somewhat limited to simple cases, and are not considered here.†

Let us now examine by means of an example the Binder-Schmidt graphical procedure, which we mentioned briefly in the preceding section.

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* See pages 16 and 93.

† See, for example, Reference 3, Chapter 15, page 225.

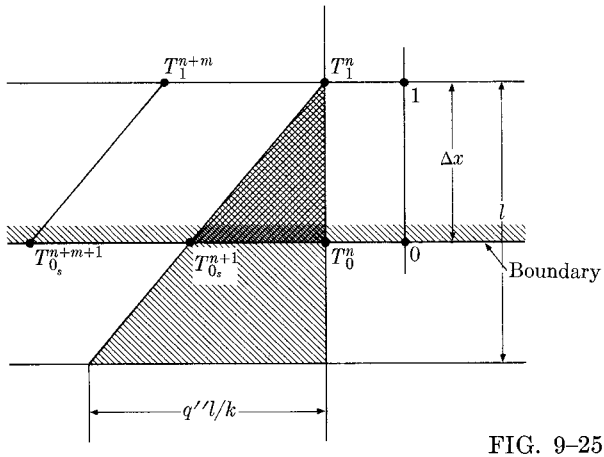


FIG. 9–25

Example 9–5. Reconsider Example 2–3, assuming the following numerical values: $L = 1$ in., $\rho = 500$ lbm/ft^3, $c = 0.1$ Btu/lbm·°F, $k = 10$ Btu/ft·hr·°F, $h = 100$ Btu/ft^2·hr·°F, and $q'' = 4800$ Btu/ft^2·hr. We wish to find the temperature rise on the upper surface two minutes after the application of heat flux to the bottom of the plate.

For purposes of illustration, let us consider a coarse network, with the plate divided into four equal layers. The temperature of the inner nodal points is obtained by taking graphically the arithmetic mean of the neighboring temperatures. As Fig. 9–20 shows, the upper boundary is taken care of by extending this boundary by an amount of $\Delta x/\Delta\text{Bi}$. Since $\Delta x = L/4 = \frac{1}{4}$ in. and $\Delta\text{Bi} = h\,\Delta x/k = 100 \times (\frac{1}{4}/12)/10 = 1/4.8$, the extension is $\Delta x/\Delta\text{Bi} = 4.8\,\Delta x$. However, so far we have developed no graphical procedure suitable to the lower boundary, which must be considered first. Since we are interested in the graphical solution only, under steady conditions we have in terms of Fig. 9–25

$$0 = k\frac{T_1^n - T_{0_s}^{n+1}}{\Delta x} + q''. \tag{9–60}$$

This may be rearranged in the form

$$\frac{T_{0_s}^{n+1} - T_1^n}{\Delta x} = \frac{q''l/k}{l}, \tag{9–61}$$

where l, an arbitrary length introduced for convenience in the graphical procedure, may be taken equal to Δx. Once the temperature gradient given by Eq. (9–61) is established by the similarity of the triangles shown in Fig. 9–25, in any time interval, say $n + m$, we need only draw a straight line from T_1^{n+m} parallel to this gradient; the intersection of the straight line with the boundary gives $T_{0_s}^{n+m+1}$.

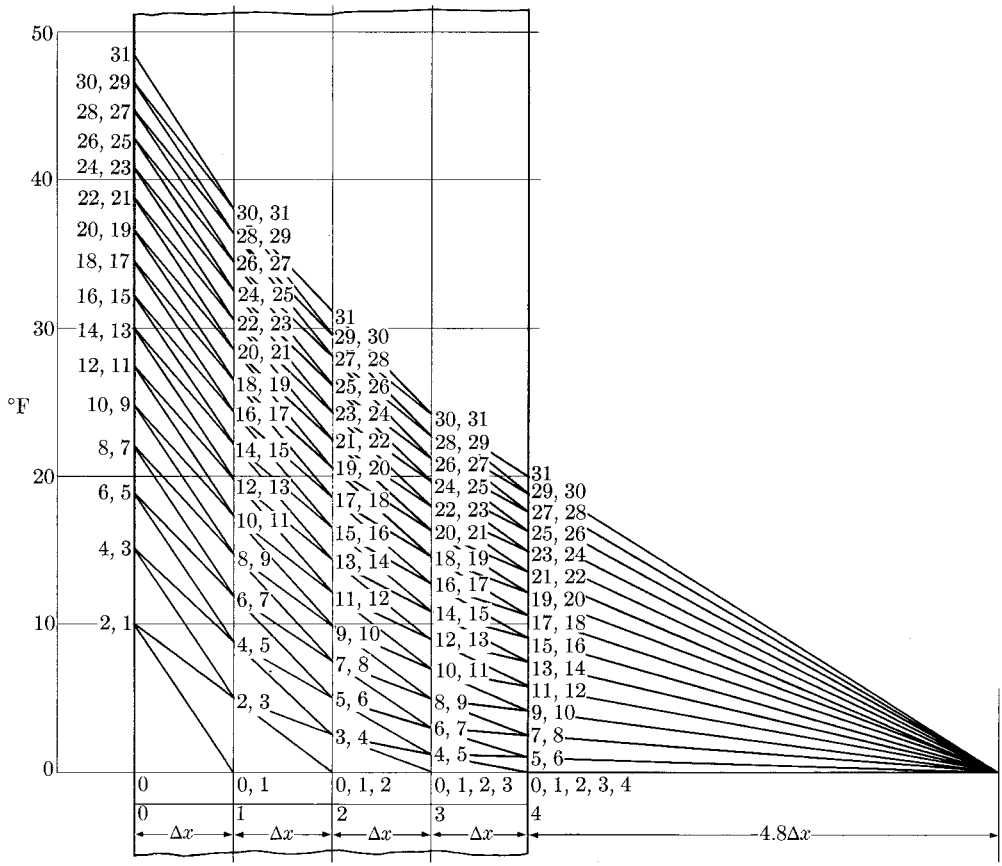


FIG. 9–26

Having completed the graphical construction suitable to the inner and to both boundary nodal points, we now proceed to the solution as indicated in Fig. 9–26. By marking the time intervals on the nodal points we clearly show the instantaneous temperature distribution associated with each interval. The temperature scale on Fig. 9–26 is obtained by rearranging Eq. (9–61) in the form

$$T_{0_s}^1 - T_1^0 = \frac{q''(\Delta x)}{k} = \frac{4800 \times 1/4}{12 \times 10} = 10°\text{F}.$$

The number n of the time intervals is found from the specified total time $t = n(\Delta t)$; since

$$a = k/\rho c = 10/(500 \times 0.1) = 0.2$$

and

$$\Delta\text{Fo} = \tfrac{1}{2} = a(\Delta t)/(\Delta x)^2$$

yield

$$\Delta t = (\Delta x)^2/2a = (\tfrac{1}{4}/12)^2 \times 3600/(2 \times 0.2) \cong 3.9 \text{ sec},$$

we have $n = 2 \times 60/3.9 \cong 31$ intervals. Figure 9–26 gives 20°F for the temperature of the upper surface at the end of 31 time intervals. (Compare the penetration time to be obtained from Fig. 9–26 with the times found in Examples 2–3 and 8–8.)

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14. G. E. Forsythe and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations*. New York: Wiley, 1960.

Problems

In the following problems, if a relaxation or step-by-step solution is to be carried out by hand or by desk calculator, a gross answer is satisfactory, and numerical values may be assigned to dimensions and properties given in symbols; if a digital computer is to be used, an accurate answer is required, depending on a parametric study in terms of the dimensionless numbers involved.

9–1. A two-dimensional extended surface made of two different materials is shown in Fig. 9–27. Find the steady temperature of the extended surface.

9–2. A Prony brake is simulated by the two-dimensional model shown in Fig. 9–28. Find the steady temperature of the brake.

9–3. An approximation of a turbine blade is shown in Fig. 9–29. Assume that the heat transfer coefficient linearly decreases from the upstream value h to the downstream value $h/2$. Find the steady temperature of the blade.

9–4. A solid cylinder rotates while pressed on another solid cylinder (Fig. 9–30). Find the steady temperature of the system.*

9–5. The hydrodynamic friction generated in the bearings of a shaft may be assumed to be a constant heat flux acting on the shaft (Fig 9–31). Find the steady temperature of the shaft.

9–6. Find the stability criterion of the explicit difference equation corresponding to the differential equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

9–7. Find the stability criterion of the explicit difference equations corresponding to the boundary conditions

$$\pm k\left(\frac{\partial T}{\partial n}\right)_\sigma + q'' = h(T_\sigma - T_\infty)$$

and

$$\pm k_1\left(\frac{\partial T_1}{\partial n}\right)_\sigma + \mu pV = \pm k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma.$$

9–8. Reconsider Problem 5–5. Assume that the length of the heat exchanger is L. (a) Obtain the typical difference equations and the associated stability criteria. (b) Find the unsteady temperature of the heat exchanger by numerical means.

9–9. A torch moving with constant velocity V applies the constant heat flux q'' to a small circular area of a flat plate (Fig. 9–32). Find the unsteady temperature of the plate by numerical means. [*Hint:* Analyze the problem by assuming that the torch is stationary and the plate is moving.]

9–10. Consider a fully developed laminar flow in a tube (Fig. 9–33). While the upstream half of the tube is insulated, the downstream half is suddenly subjected to (i) a constant heat flux q'', (ii) a periodic heat flux $q_0''(1 + \epsilon \sin \omega t)$. The axial conduction is appreciable both in the fluid and tube-walls. (a) Obtain the typical difference equations and the associated stability criteria. (b) Show the effect of axial conduction, tube material, and unsteadiness on the distribution of the steady heat transfer coefficient.

9–11. Consider a flat plate that is initially at a uniform temperature (Fig. 9–34). Assume that the constant internal energy u''' begins to be generated in the plate. (a) Modify the graphical procedure by including the effect of energy generation.

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* This problem corresponds to part (f) of Problem 4–29.

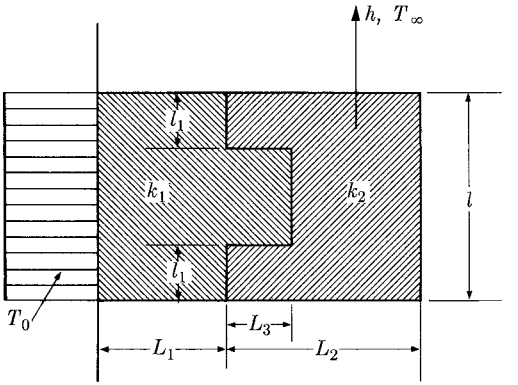


FIG. 9–27

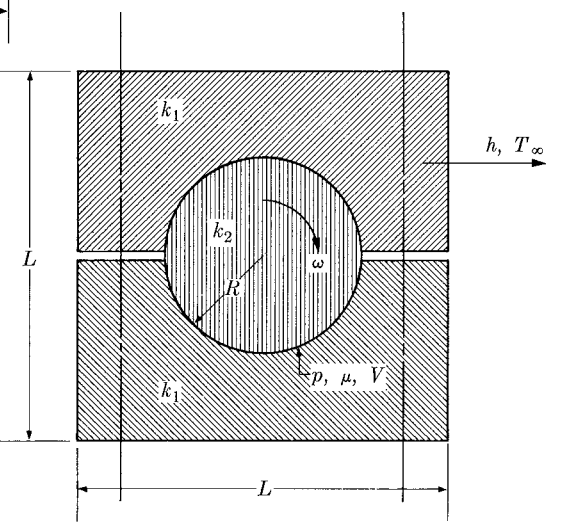


FIG. 9–28

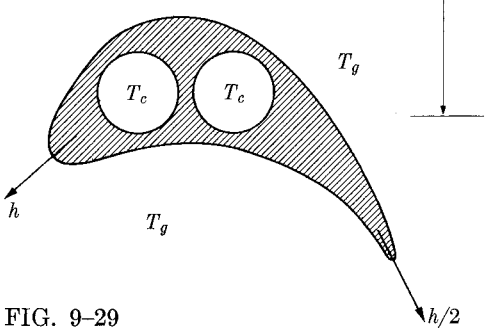


FIG. 9–29

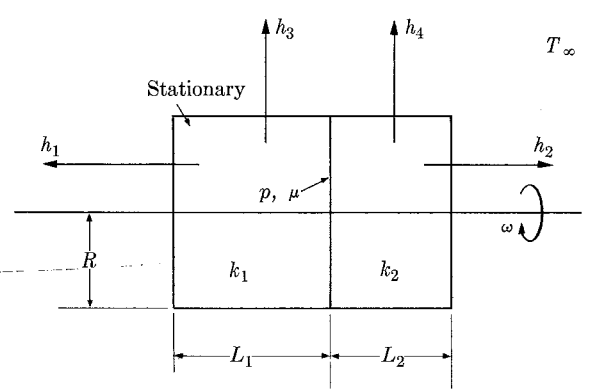


FIG. 9–30

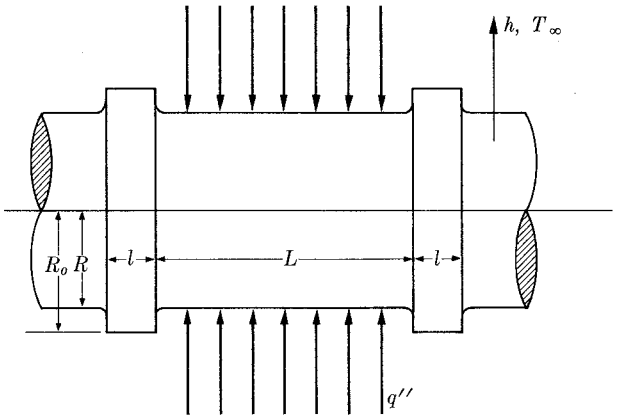


FIG. 9–31

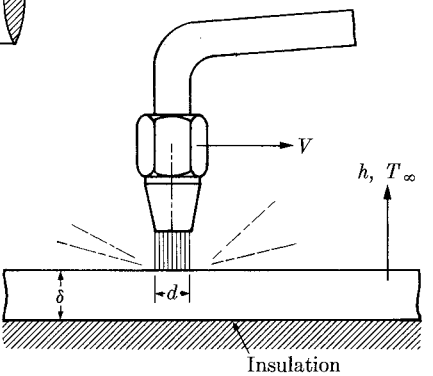


FIG. 9–32

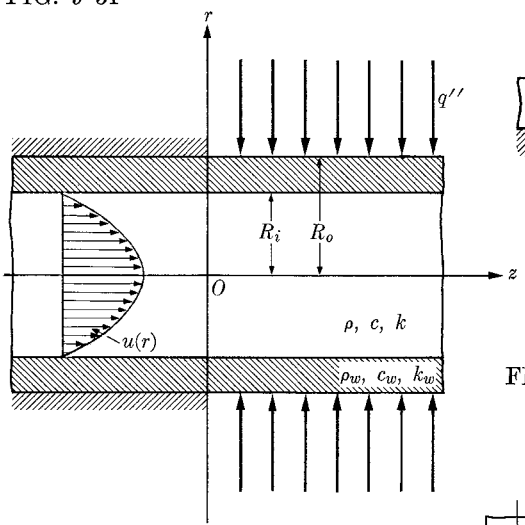


FIG. 9–33

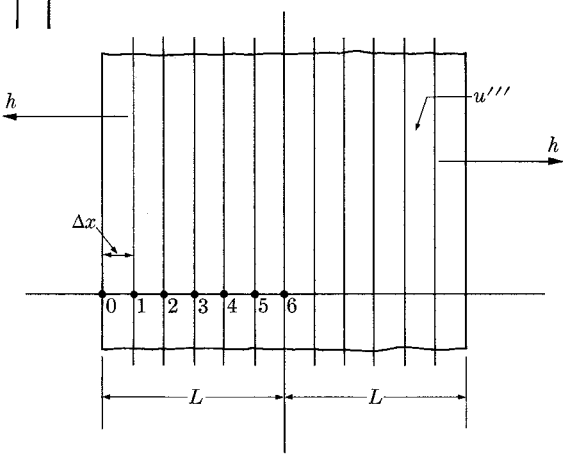


FIG. 9–34

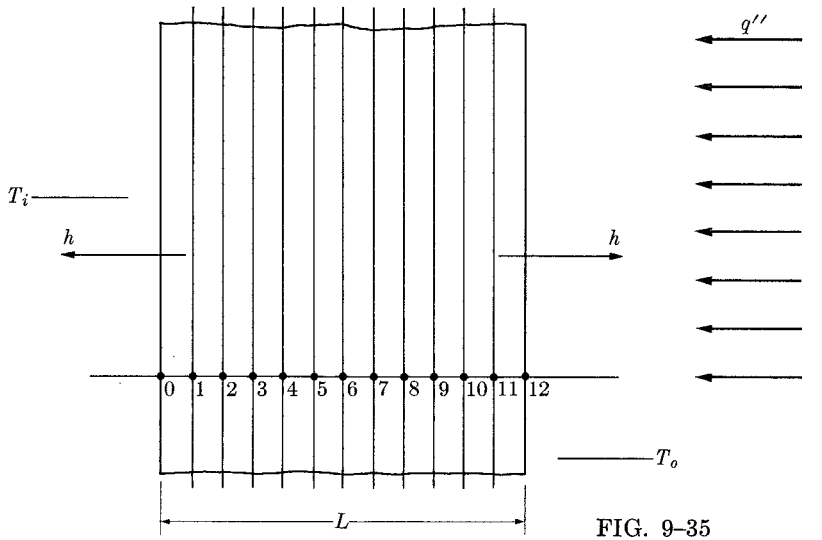


FIG. 9–35

(b) Using part (a), find the temperature distribution in the plate after 10 min have elapsed. Data:

$$L = 6 \text{ in.}, \quad \rho = 480 \text{ lbm/ft}^3, \quad c = 0.1 \text{ Btu/lbm·°F},$$
$$k = 10 \text{ Btu/ft·hr·°F}, \quad h = 100 \text{ Btu/ft}^2\text{·hr·°F}, \quad u''' = 14{,}400 \text{ Btu/ft}^3\text{·hr}.$$

9–12. Develop graphical procedures suitable to cylindric and spherical geometry.

9–13. Consider the brick walls of a house. The inside and outside heat transfer coefficients are the same. The inside temperature T_i is greater than the outside temperature T_o. There is a steady heat loss from the house to the atmosphere. Now let us assume that we start collecting solar radiation q'' on the outer surface of the walls (Fig. 9–35). Using the graphical procedure, determine the time required to stop the heat loss from the house. Data:

$$L = 1 \text{ ft}, \quad \rho = 100 \text{ lbm/ft}^3, \quad c = 0.2 \text{ Btu/lbm·°F},$$
$$k = 0.4 \text{ Btu/ft·hr·°F}, \quad h = 2 \text{ Btu/ft}^2\text{·hr·°F},$$
$$q'' = 440 \text{ Btu/ft}^2\text{·hr}, \quad T_i = 75°\text{F}, \quad T_o = -30°\text{F}.$$

9–14. Reconsider the distributed case of Problem 6–2. Using the graphical procedure, carry out the temperature variation of the plate for the first twenty intervals of time. Data:

$$\delta = \tfrac{1}{2} \text{ in.}, \quad \Delta x = \delta/12, \quad \rho = 500 \text{ lbm/ft}^3,$$
$$c = 0.1 \text{ Btu/lbm·°F}, \quad k = 10 \text{ Btu/ft·hr·°F},$$
$$h = 2000 \text{ Btu/ft}^2\text{·hr·°F}, \quad T_o = 100°\text{F}, \quad \omega = 5 \text{ min}^{-1}.$$

9–15. Reconsider Problem 2–9. Assume that the heat transfer coefficient and all properties are constant except the thermal conductivity of the insulation, which varies in the form

$$k_i = k_\infty(1 + \beta \Delta T),$$

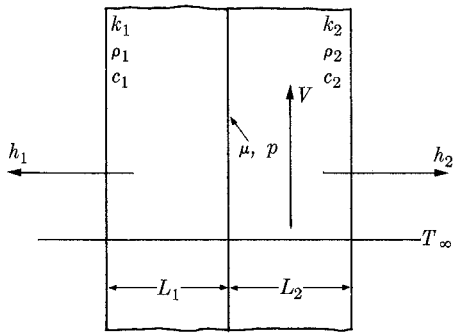


FIG. 9–36

where ΔT is measured relative to the ambient temperature. Using the graphical procedure, carry out the temperature variation of the system for the first twenty intervals of time. Data:

$$\delta = \tfrac{1}{16} \text{ in.}, \quad L = \tfrac{1}{2} \text{ in.}, \quad \Delta x = L/8,$$
$$k_\infty = 0.1 \text{ Btu/ft·hr·°F}, \quad \beta = 0.01,$$
$$k_h = 200 \text{ Btu/ft·hr·°F}, \quad a_i = 0.01 \text{ ft}^2\text{/hr},$$
$$a_h = 4 \text{ ft}^2\text{/hr}, \quad h = 2 \text{ Btu/ft}^2\text{·hr·°F},$$
$$u''' = 148{,}000 \text{ Btu/ft}^3\text{·hr}.$$

9–16. Simulate a dry friction brake by two flat plates (Fig. 9–36). The initial temperature of the brake is equal to that of the ambient, T_∞. Then one of the plates suddenly assumes a constant velocity V. The coefficient of dry friction and the pressure between the plates are μ and p, respectively. Using the graphical procedure, carry out the temperature variation of the brake for the first twenty intervals of time. Data:

$$L_1 = L_2 = \tfrac{1}{2} \text{ in.}, \quad \mu pV = 144{,}000 \text{ Btu/ft}^2\text{·hr},$$
$$\rho_1 = \rho_2 = 500 \text{ lbm/ft}^3, \quad c_1 = c_2 = 0.1 \text{ Btu/lbm·°F},$$
$$k_1 = 20 \text{ Btu/ft·hr·°F}, \quad k_2 = 30 \text{ Btu/ft·hr·°F},$$
$$h_1 = 400 \text{ Btu/ft}^2\text{·hr·°F}, \quad h_2 = 600 \text{ Btu/ft}^2\text{·hr·°F}.$$

DIFFERENTIAL-DIFFERENCE FORMULATION. ANALOG SOLUTION

The final method of solution to be considered in this text is analogy. A number of lumped and distributed models for conduction problems are available, based on mechanical, hydrodynamic, and electrical systems. Networks of electrical resistors, capacitors, and sometimes inductors are the most important simulators of lumped systems; on rare occasions, mechanical systems comprised of masses, springs, and dashpots are also used for this purpose. Electrolytic tanks, conductive paper, stretched membranes, soap film, fluid mappers, and polarized light are some of the distributed models occasionally used. Here, because of its versatility, we shall make use of the electrical analogy.

10–1. Analogy Between Conduction and Electricity

Let us first formulate the diffusion of electricity through a resistive and capacitive solid.† We illustrate the point in terms of one-dimensional cartesian geometry. Following the steps of formulation given in Section 2–9, we select the system shown in Fig. 10–1. Applying the general law of electricity, *the conservation of electric charge* q (coulomb), to this system yields‡

$$\partial \rho_e / \partial t = - \partial j_x / \partial x, \qquad (10\text{–}1)$$

where $\rho_e(\text{coulomb}/L^3) = q/A(\Delta x)$ is the charge density and $j_x(\text{ampere}/L^2) = i/A$ is the electric current density which, by definition, is the charge flux (diffusion of charge per unit time and area); i(ampere) is the electric current or diffusion of charge per unit time,

$$i = dq/dt. \qquad (10\text{–}2)$$

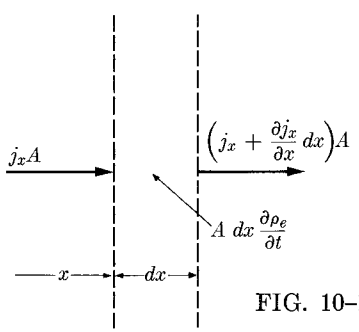


FIG. 10–1

† A practical example of such a solid would be thin layers of conducting and dielectric materials composed parallel in the direction of current.

‡ The general differential form of Eq. (10–1) is

$$\partial \rho_e / \partial t + \nabla \cdot \mathbf{j} = 0,$$

and the corresponding integral formulation is

$$\int_{\mathcal{V}} (\partial \rho_e / \partial t)\, d\mathcal{V} + \int_{\sigma} \mathbf{j} \cdot \mathbf{n}\, d\sigma = 0.$$

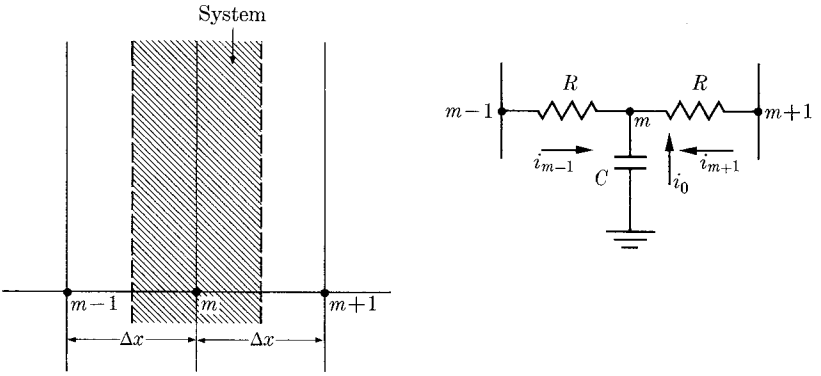


FIG. 10–2

Let us express Eq. (10–1) in terms of the electric potential E(volt) by means of the particular laws†

$$\rho_e = C^*E, \qquad \text{Coulomb's law,} \tag{10–3}$$

$$j_x = -\frac{1}{R^*}\frac{\partial E}{\partial x}, \qquad \text{Ohm's law,} \tag{10–4}$$

where C^*(farad/L^3) is the capacity per unit volume and R^*(ohm/L) the resistivity per unit length of the solid. Thus we obtain as the governing equation for the one-dimensional diffusion of electricity‡

$$R^*C^*\frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2}, \tag{10–5}$$

which is analogous to that for the one-dimensional conduction of heat

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{10–6}$$

Therefore, the solution of Eq. (10–6) may be found by solving Eq. (10–5), that is, by measuring the diffusion of electricity in a resistive-capacitive solid. Furthermore, expressing the space derivative of Eq. (10–5) in difference form (Fig. 10–2a), and subsequently rearranging we have

$$R^*C^*(\Delta x)^2\frac{dE_m}{dt} = E_{m+1} + E_{m-1} - 2E_m, \tag{10–7}$$

which is the *timewise differential, spacewise difference formulation* of the problem.

The important fact is that a formulation of this type can be simulated by a lumped circuit element consisting of one capacitor and two resistors as shown

† Discuss the meaning and equivalents of Eqs. (10–1), (10–3), and (10–4) in conduction.
‡ This equation becomes the one-dimensional electromagnetic wave equation when the solid is further assumed to have inductance.

in Fig. 10–2(b). The governing equation of this circuit is identical to Eq. (10–7), provided R and C are the total resistance and capacitance of the system, respectively, so that

$$R(\text{ohm}) = R^*(\Delta x)/A \qquad \text{and} \qquad C(\text{farad}) = C^*A(\Delta x). \tag{10–8}$$

To show this fact we apply the steps of formulation to Fig. 10–2(b). The system of the distributed case now reduces to nodal point m, and the general law given by Eq. (10–1) becomes *Kirchhoff's current law* applied to this point. It follows then that

$$i_{m-1} + i_0 + i_{m+1} = 0. \tag{10–9}$$

Furthermore, for convenience the particular laws may now be written in terms of charge and current rather than in terms of their densities. Thus Coulomb's and Ohm's laws, the former rearranged by Eq. (10–2) and the latter expressed in terms of the potential difference, become

$$i = C\frac{dE}{dt}, \tag{10–10}$$

and

$$i = \frac{\Delta E}{R}, \tag{10–11}$$

respectively. Interpreting Eqs. (10–10) and (10–11) in terms of Fig. 10–2(b), and inserting the result into Eq. (10–9) gives the governing equation

$$RC\frac{dE_m}{dt} = E_{m+1} + E_{m-1} - 2E_m, \tag{10–12}$$

which is identical to Eq. (10–7), as may readily be shown by means of Eq. (10–8). Thus we prove that the circuit element given by Fig. 10–2(b) is the approximate *analog solution* of the one-dimensional heat conduction governed by Eq. (10–6). Furthermore, since the differential-difference formulation that corresponds to Eq. (10–6) is

$$\frac{(\Delta x)^2}{a}\frac{dT_m}{dt} = T_{m+1} + T_{m-1} - 2T_m, \tag{10–13}$$

the comparison between Eqs. (10–12) and (10–13) yields that if temperatures are made proportional to potentials, then $(\Delta x)^2/a$ becomes proportional to RC.

Inspection of Eq. (10–12) reveals that the analog solution of two-dimensional conduction is governed by the equation

$$RC\frac{dE_{m,n}}{dt} = E_{m-1,n} + E_{m,n-1} + E_{m+1,n} + E_{m,n+1} - 4E_{m,n}, \tag{10–14}$$

which is satisfied by the circuit element shown in Fig. 10–3. The extension of the procedure to three-dimensional cartesian geometry, moving solids, boundary

conditions, and other geometry presents no difficulty. This is left to the reader as an exercise (see Problems 10–1 and 10–2).

From Eqs. (10–12) and (10–14) we may generalize the fact that the approximate analog solution of conduction problems requires a circuit capable of performing multiplication by a constant, addition, integration, and differentiation. Electric circuits that can accomplish these operations are called *passive* circuits if they include, like those in Figs. 10–2(b) and 10–3, only fixed resistors and capacitors, and possibly inductors and transformers. They are called *active* circuits if, like electronic amplifiers, they involve additional elements drawing energy from an external source. Let us now examine the operations mentioned above as performed by passive circuit elements.

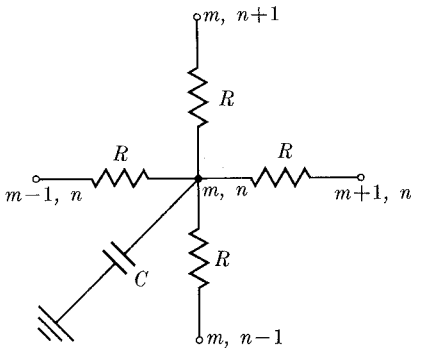
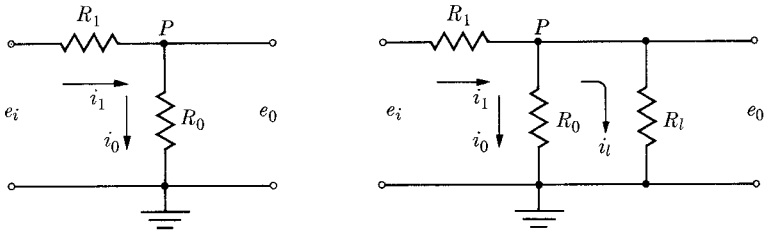


FIG. 10–3



(a) No load (b) With load

FIG. 10–4

10–2. Passive Circuit Elements

We start with multiplication, which can be accomplished by the resistive circuit element shown in Fig. 10–4(a). Kirchhoff's current law applied to nodal point P yields

$$i_1 - i_0 = 0.$$

Expressing the currents in terms of potentials by the use of Ohm's law, we obtain

$$\frac{e_i - e_0}{R_1} - \frac{e_0}{R_0} = 0,$$

which may be rearranged in the form

$$e_0 = \left(\frac{R_0}{R_0 + R_1}\right) e_i. \tag{10–15}$$

Thus the circuit element of Fig. 10–4(a) multiplies the potential e_i with the positive constant $R_0/(R_0 + R_1)$. However, when this element is used in a circuit its output is, in general, *loaded*. Let us now see how Eq. (10–15) is affected by the resistance R_l equivalent to the output load (Fig. 10–4b). Kirchhoff's current law applied to nodal point P now becomes

$$i_1 - i_0 - i_l = 0,$$

and relating the currents to potentials by Ohm's law we obtain

$$\frac{e_i - e_0}{R_1} - \frac{e_0}{R_0} - \frac{e_0}{R_l} = 0,$$

which may be rearranged to give

$$e_0 = \left(\frac{R_0}{R_0 + R_1 + R_0 R_1/R_l}\right) e_i. \tag{10–16}$$

Note that Eq. (10–16) approaches Eq. (10–15) as $R_l \to \infty$ only; in any other circumstances the output potential is affected by the loading. Since we are interested in multiplying the input potential by a predetermined positive constant, the deviation of the output potential from Eq. (10–15) is called the *error* of the circuit element due to loading.

Next we consider integration, which may be approximately achieved by the circuit shown in Fig. 10–5. Again we write Kirchhoff's current law for nodal point P,

$$i_1 - i_0 = 0.$$

Relating i_0 and i_1 to the potentials by Ohm's and Coulomb's laws, respectively, gives

$$\frac{e_i - e_0}{R} + C\frac{de_0}{dt} = 0.$$

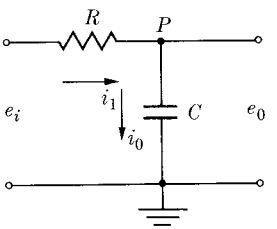


FIG. 10–5

Now, if we assume that $e_i \gg e_0$, it follows that

$$e_0 = \frac{1}{RC}\int_0^t e_i\, d\tau, \tag{10–17}$$

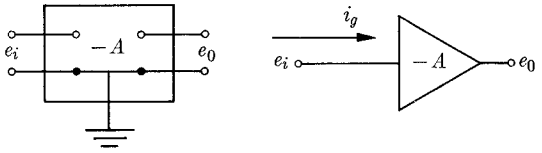
which is the integral of the input potential. Note that the circuit element for integration is *approximate* even in the absence of any load because of the assumption $e_i \gg e_0$. The effect of loading may be determined by a procedure similar

to that followed for the preceding circuit element. This is left to the reader (see Problem 10–4).

It is clear from the foregoing discussion that all passive circuit elements suffer from loading errors. In general, these elements operate correctly only if the impedance at the output of the circuit element is very high. Since active circuit elements such as electronic amplifiers have input impedances often in excess of several megohms, the coupling of a passive circuit element with an amplifier eliminates the loading error on the element. Of course, the use of amplifiers increases both the first cost and the operating cost of the analog circuit (hereafter to be referred to as the *analog computer*); however, this expense may be well justified by the increased accuracy of the solution. For most general-purpose analog computers, the high-gain DC amplifier discussed in the next section is employed almost exclusively. In general, AC analog computers are less expensive to build and require simpler auxiliaries; however, the phase shifts in AC circuits tend to make this type of computer less accurate than DC computers.

10–3. Active Circuit Elements. High-Gain DC Amplifiers

The most commonly employed active elements are high-gain DC amplifiers, function generators, and function multiplier-dividers. We shall consider only amplifiers.



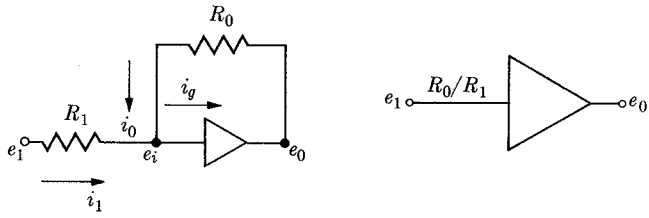
(a) Schematic (b) Symbolic FIG. 10–6

The design of amplifiers and the function of its components are beyond the scope of the text. Interested readers may refer to the references at the end of this chapter. Here we picture the amplifier as simply a black box (Fig. 10–6), and discuss only those amplifier characteristics conceptually important for analog computer solutions. Designating the input and output potentials relative to the ground by e_i and e_0, respectively, we first define the gain A of the amplifier by the ratio†

$$A = -e_0/e_i.$$

The basic requirement of the amplifier is that it have as high a gain as possible. The practical range of this gain is 10^4–10^8. Assuming that for the purpose of recording, the output potential is to be kept between ± 100 volts, we obtain

† The sign of this ratio will be discussed in connection with Fig. 10–7.



(a) Schematic (b) Symbolic

FIG. 10–7

a maximum possible value of 0.01 volts for the input potential. Since this potential is usually connected to a vacuum-tube grid which has an extremely high impedance, the input current is very small. Hence the basic and associated properties required from the amplifier as an analog component are: (i) a very high gain A, (ii) approximately zero input potential e_i, (iii) negligibly small input current i_g. Of course, there are other important properties required from the amplifier, such as linear output over a wide range, flat frequency response, low noise level, etc. These, however, are means to further improvement in the accuracy of the solution, rather than basic requirements for the use of the amplifier.

Let us now consider a circuit consisting of a high-gain DC amplifier, an input resistor R_1, and a feedback resistor R_0 (Fig. 10–7). A potential, say e_1, is applied to the input of the amplifier through the resistor R_1. Since $i_g = 0$ according to property (iii), Kirchhoff's law applied to the input of the amplifier gives

$$i_1 + i_0 = 0.$$

Then, relating currents to potentials by Ohm's law, and noting that $e_i = 0$ because of property (ii), we obtain

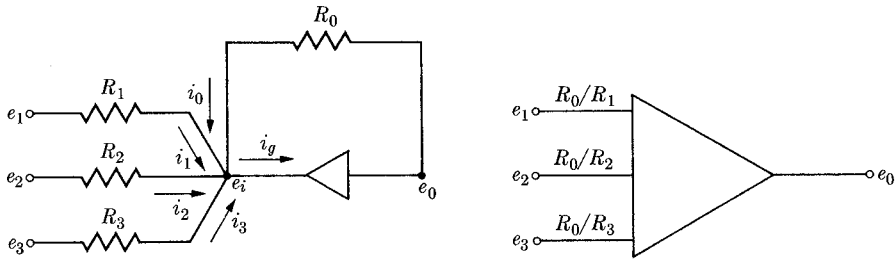
$$e_1/R_1 + e_0/R_0 = 0,$$

which may be rearranged in the form

$$e_0 = -\frac{R_0}{R_1} e_1. \tag{10–18}$$

Thus we have shown that the circuit element given by Fig. 10–7 is a *multiplier*. Clearly, the same circuit can be used as a *sign changer* by taking the ratio R_0/R_1 equal to unity.

Note that in analog computer applications, the DC amplifier always includes a feedback circuit as exemplified by Fig. 10–7. It is for this reason that the amplifier gain A has to be negative. For, if it were positive, an increase at the output potential would increase the input potential which, in turn, would further increase the output potential, causing instability. The sign of amplifier gain is made negative by using an odd number of amplification stages. This is



(a) Schematic (b) Symbolic

FIG. 10–8

a problem of electronics, however, and is not our concern. Note, moreover, that this sign has nothing to do with that of Eq. (10–18).

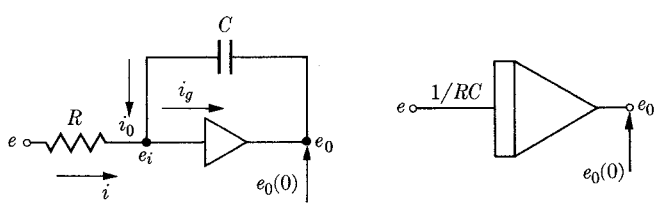
Let us now reconsider the circuit element given by Fig. 10–7 and replace the input resistor by three parallel resistors as shown in Fig. 10–8. Again, Kirchhoff's law applied to the input of the amplifier in the light of property (iii), and rearranged by Ohm's law and property (ii) gives

$$e_1/R_1 + e_2/R_2 + e_3/R_3 + e_0/R_0 = 0,$$

which yields

$$e_0 = -\left(\frac{R_0}{R_1}e_1 + \frac{R_0}{R_2}e_2 + \frac{R_0}{R_3}e_3\right). \tag{10–19}$$

We learn from Eq. (10–19) that the circuit element of Fig. 10–8 can be used as an *adder*, as clearly seen from the special case $R_0/R_1 = R_0/R_2 = R_0/R_3 = 1$.



(a) Schematic (b) Symbolic

FIG. 10–9

We now return once more to the circuit element given by Fig. 10–7, and this time replace the feedback element by a capacitor C as indicated in Fig. 10–9. Kirchhoff's law applied to the input of the amplifier and rearranged by Ohm's and Coulomb's laws then gives

$$\frac{e}{R} + C\frac{de_0}{dt} = 0$$

which, by integration, may be rearranged in the form

$$e_0 = -\frac{1}{RC}\int_0^t e\,d\tau + e_0(0), \tag{10–20}$$

where $e_0(0)$ is the initial condition, the potential to which the capacitor is charged at $t = 0$. Equation (10–20) indicates that the circuit element of Fig. 10–9 can be employed as an *integrator*.

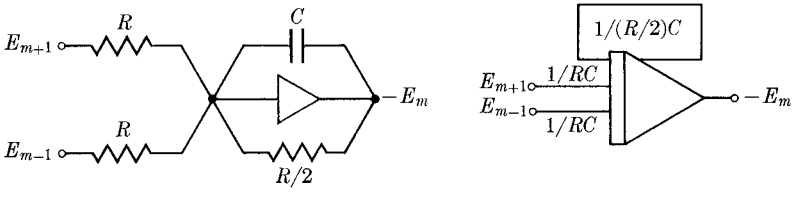
Clearly, a DC amplifier may also be used as a *differentiator*. (Draw the corresponding circuit.) However, whenever possible this operation should be avoided because of noise.

Thus we have seen the use of the DC amplifier as a multiplier, a sign changer, an adder, and an integrator. The two other important active circuit elements, function generators and function multiplier-dividers, require a somewhat lengthy and special treatment, and will not be considered here. The interested reader may consult the references cited at the end of this chapter.

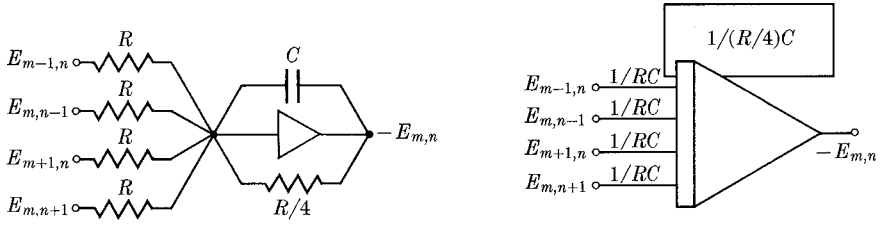
We now proceed to the use of DC amplifiers in the solution of conduction problems. We begin by considering the active element corresponding to the passive element shown in Fig. 10–2 and governed by Eq. (10–12). If this equation is rearranged in the form

$$\frac{E_{m+1}}{R} + \frac{E_{m-1}}{R} + \frac{-E_m}{R/2} + C\frac{d(-E_m)}{dt} = 0, \tag{10–21}$$

then the desired active circuit element is one that involves a DC amplifier whose input nodal point satisfies Kirchhoff's current law corresponding to Eq. (10–21) (Fig. 10–10). The extension of the foregoing procedure to two-dimensional geometry follows immediately. Thus we obtain the active element (Fig. 10–11)



(a) Schematic FIG. 10–10 (b) Symbolic



(a) Schematic FIG. 10–11 (b) Symbolic

corresponding to the passive element given in Fig. 10–3 and governed by Eq. (10–14). Further extension to three-dimensional cartesian geometry, moving solids, boundary conditions, and other geometries presents no difficulty. This is left to the reader as an exercise.

10–4. Examples

For further experience in analog solutions we next consider a number of lumped and distributed problems. Since active circuits involve negligible error as compared with passive circuits, they are more commonly used in practice; hence we shall discuss the solution of these problems in terms of active circuits only.

Example 10–1. One surface of a flat plate is suddenly subjected to constant heat flux q''; the other surface is insulated (Fig. 10–12). The thickness of the plate is δ, and the initial temperature T_0. We wish to find the analog solution of the problem.

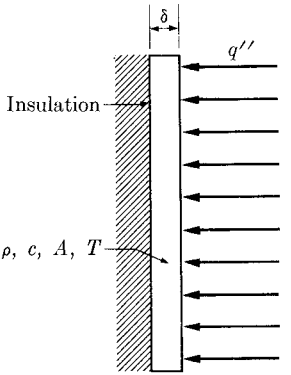


FIG. 10–12

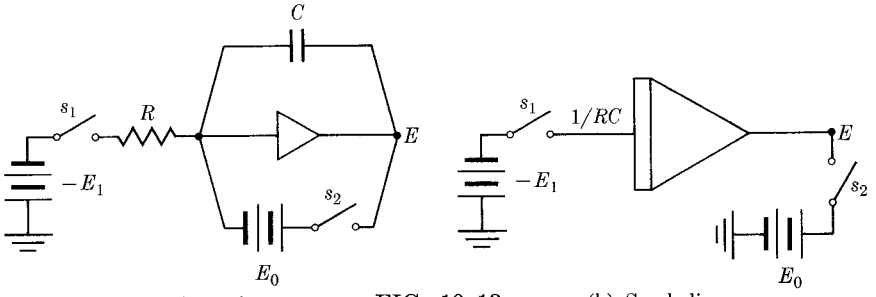
Assuming that the temperature of the plate may be lumped, we find the formulation of the problem to be

$$dT/dt - m = 0, \qquad T(0) = T_0, \qquad (10\text{–}22)$$

where $m = q''/\rho c\delta$. Equation (10–22) can readily be simulated by

$$C\frac{dE}{dt} + \frac{(-E_1)}{R} = 0, \qquad E(0) = E_0. \qquad (10\text{–}23)$$

The desired analog solution is obtained from an active circuit that includes a DC amplifier whose input nodal point satisfies Kirchhoff's current law corresponding to Eq. (10–23) (Fig. 10–13). If E is made proportional to T, consequently E_0 to T_0, the values of E_1, R, and C must be so adjusted that E_1/RC is proportional to m in the same sense. The circuit operates as follows: two switches, both possibly activated by the same relay coil, are employed to apply



(a) Schematic FIG. 10–13 (b) Symbolic

the initial condition and driving potential; initially S_1 is open and S_2 is closed; suddenly S_1 is closed and S_2 is opened.

Example 10–2. A square plate of thickness δ is pressed on a solid surface by force P (Fig. 10–14). The plate suddenly assumes constant velocity V on the surface. The coefficient of dry friction between the plate and the solid is μ. The conductivity of the solid is much smaller than that of the plate. Initially the plate is at ambient temperature T_∞. The heat transfer coefficient is h. We wish to find the analog solution of the problem.

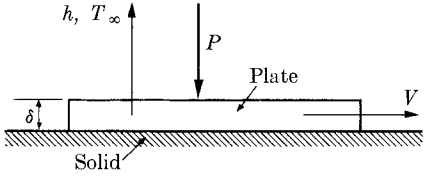


FIG. 10–14

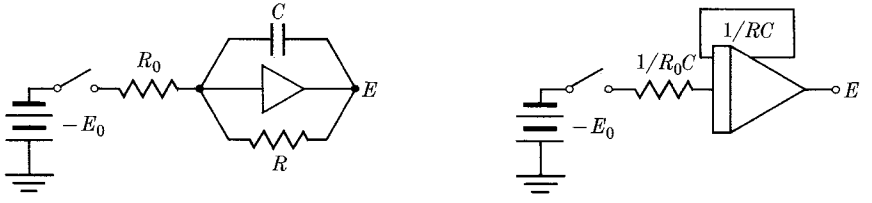
Assuming that the plate temperature may be lumped, and neglecting the heat transfer to the solid, we find that the formulation of the problem in terms of $\theta = T - T_\infty$ is

$$d\theta/dt + m\theta = n, \qquad \theta(0) = 0, \tag{10–24}$$

where $m = h/\rho c\delta$ and $n = \mu PV/\rho c\delta A$. The corresponding analog formulation is

$$C\frac{dE}{dt} + \frac{E}{R} + \frac{(-E_0)}{R_0} = 0. \tag{10–25}$$

An active circuit which satisfies Eq. (10–25) at the inlet nodal point of the DC amplifier provides the required analog solution (Fig. 10–15). The values of R_0, R, and C are selected by considering the proportionality of m to $1/RC$ and that of n to E_0/R_0C.



(a) Schematic (b) Symbolic

FIG. 10–15

Example 10–3. Let us find the analog solution of Problem 5–3.

The solution of this problem by means of Laplace transforms was discussed in Example 7–2. Referring to that example, and after some rearrangements,

we write the formulation of the problem in the form†

$$\frac{d\psi}{dt} + b_1\psi - b_1\phi = 0, \tag{10–26}$$

$$\frac{d\phi}{dt} + (b_2 + b_3)\phi - b_2\psi = 0, \tag{10–27}$$

$$\psi(0) = \phi(0) = \psi_0. \tag{10–28}$$

Simulating ψ by E and ϕ by e, we find that the corresponding analog equations are

$$C_1\frac{dE}{dt} + \frac{E}{R_1} + \frac{(-e)}{R_1} = 0, \tag{10–29}$$

$$C_2\frac{de}{dt} + \frac{e}{R_3} + \frac{(-E)}{R_2} = 0, \tag{10–30}$$

$$E(0) = e(0) = E_0. \tag{10–31}$$

Although the first two terms of Eqs. (10–29) and (10–30) are easily satisfied at the input of two DC amplifiers, the third term of each equation requires the reverse value of the potential associated with the first two terms of the other equation. Thus the use of another amplifier as a sign changer at the output of each of the first two amplifiers becomes necessary. The analog solution of the problem is shown in Fig. 10–16. However, by simply changing the sign of Eq. (10–29) or Eq. (10–30), we can eliminate the amplifiers employed as sign changers; multiplying one of these equations, say the latter, by -1 gives

$$C_2\frac{d(-e)}{dt} + \frac{(-e)}{R_3} + \frac{E}{R_2} = 0. \tag{10–32}$$

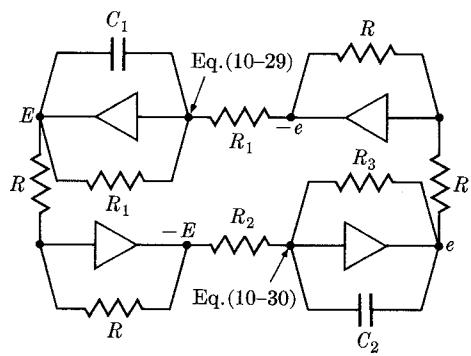
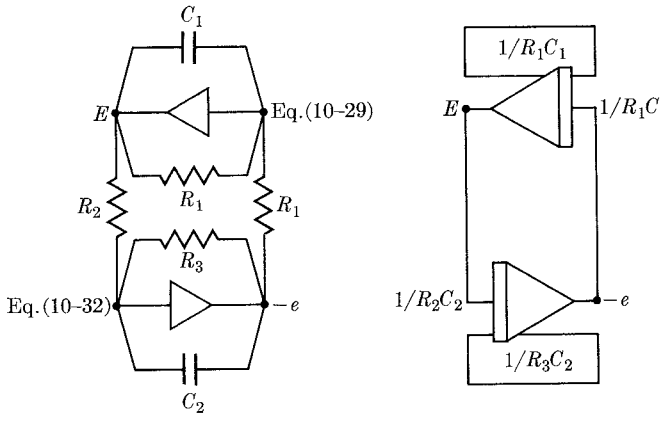


FIG. 10–16

† See Eqs. (7–29), (7–30), and (7–31).



(a) Schematic (b) Symbolic FIG. 10–17

Now we need only E and $-e$. In fact, the circuit indicated in Fig. 10–17 satisfies Eqs. (10–29) and (10–32). Note that both capacitors must initially be loaded according to Eq. (10–31). Also, the values of R_1, R_2, R_3, C_1, and C_2 must be so selected that b_1 is proportional to $1/R_1C_1$, $b_2 + b_3$ to $1/R_3C_2$, and b_2 to $1/R_2C_2$.

Example 10–4. Let us find the analog solution of Example 9–5.

Since again our purpose is demonstration rather than an accurate solution, let us use the coarse subdivision considered in Example 9–5. For the typical inner nodal point we may use the circuit element in Fig. 10–10; however, we need to develop the circuit elements corresponding to boundary nodal points.

In terms of Fig. 9–25, the differential-difference formulation of the boundary associated with the imposed heat flux is

$$\frac{(\Delta x)^2}{2a}\frac{dT_0}{dt} = T_1 - T_0 + \frac{q''(\Delta x)}{k}. \tag{10–33}$$

Since $(\Delta x)^2/a$ has been made proportional to RC in connection with our discussion of the inner nodal points,† the simulation of Eq. (10–33) gives

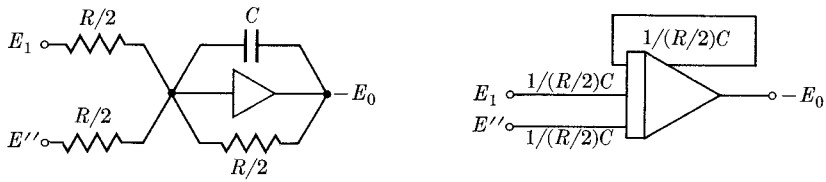
$$\frac{E_1}{R/2} + \frac{E''}{R/2} + \frac{(-E_0)}{R/2} + C\frac{d(-E_0)}{dt} = 0, \tag{10–34}$$

where E'' is proportional to $q''(\Delta x)/k$. The circuit element that satisfies Eq. (10–34) is given in Fig. 10–18.

In terms of Fig. 9–20 and its rearranged nomenclature, the differential-difference formulation of the boundary that transfers heat to the ambient is

$$\frac{(\Delta x)^2}{2a}\frac{dT_4}{dt} = T_3 - T_4 + \Delta\mathrm{Bi}\,(T_\infty - T_4). \tag{10–35}$$

† Recall the comparison between Eqs. (10–12) and (10–13).



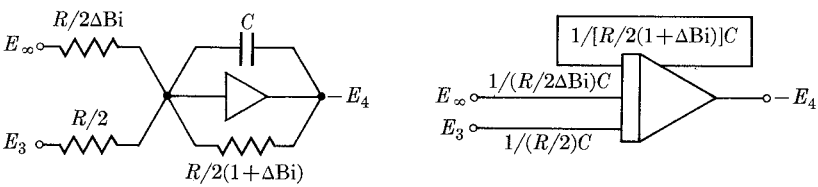
(a) Schematic (b) Symbolic

FIG. 10–18

The simulation of Eq. (10–35) yields

$$\frac{E_3}{R/2} + \frac{E_\infty}{R/2\Delta\text{Bi}} + \frac{(-E_4)}{R/2(1+\Delta\text{Bi})} + C\frac{d(-E_4)}{dt} = 0, \qquad (10\text{–}36)$$

where E_∞ is proportional to T_∞. The circuit element indicated in Fig. 10–19 satisfies Eq. (10–36).

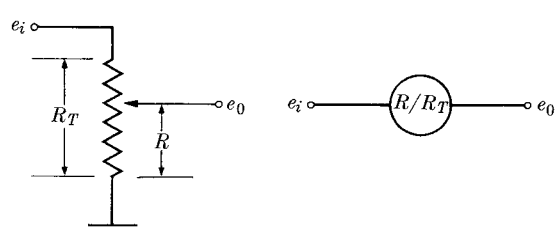


(a) Schematic (b) Symbolic

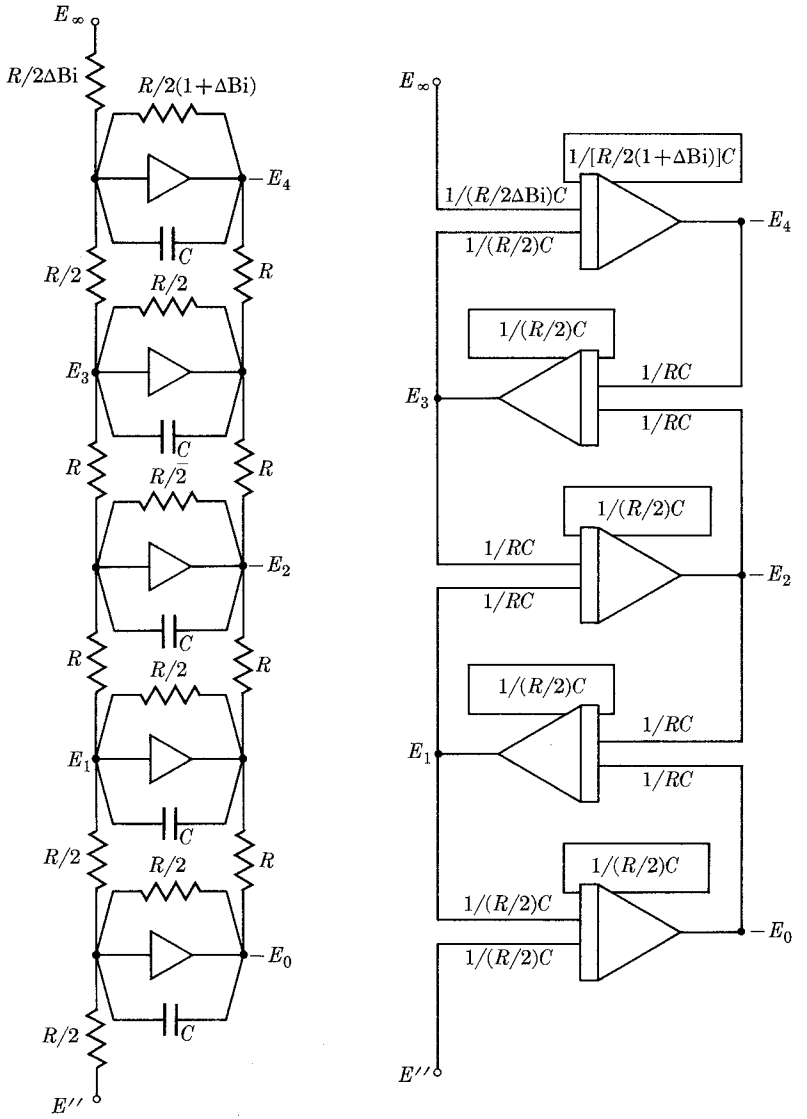
FIG. 10–19

If the numerical value of ΔBi requires a fraction of an available resistance, any necessary adjustment can be made by means of a potentiometer (Fig. 10–20).

When the appropriate circuit elements are assigned to nodal points, the existence of both plus and minus potentials requires the use of additional amplifiers as sign changers. However, these amplifiers can be eliminated as in the preceding example, by simply multiplying the potentials of alternate circuit elements by −1. The analog solution thus obtained is shown in Fig. 10–21.



(a) Schematic (b) Symbolic FIG. 10–20



(a) Schematic (b) Symbolic

FIG. 10–21

The circuit operates as follows: initially the input and output potentials have the same value E_∞, which may conveniently be assumed zero; then the input potential is suddenly changed to E''.

Example 10–5. Let us find the circuit element associated with Example 9–4. In terms of Fig. 9–23 the differential-difference formulation that corresponds to

the difference formulation given in Example 9–4 is

$$\rho cA(\Delta x)\frac{dT_m}{dt} = \rho cAV(T_{m-1} - T_m) + h_iP_i(\Delta x)(\theta_m - T_m),$$

$$\rho'c'A'(\Delta x)\frac{d\theta_m}{dt} = h_iP_i(\Delta x)(T_m - \theta_m) + h_oP_o(\Delta x)(0 - \theta_m),$$

which may be rearranged in the form

$$\frac{dT_m}{dt} = \frac{V}{\Delta x}T_{m-1} + \left(\frac{V}{\Delta x} + \frac{h_iP_i}{\rho cA}\right)(-T_m) + \frac{h_iP_i}{\rho cA}\theta_m, \tag{10–37}$$

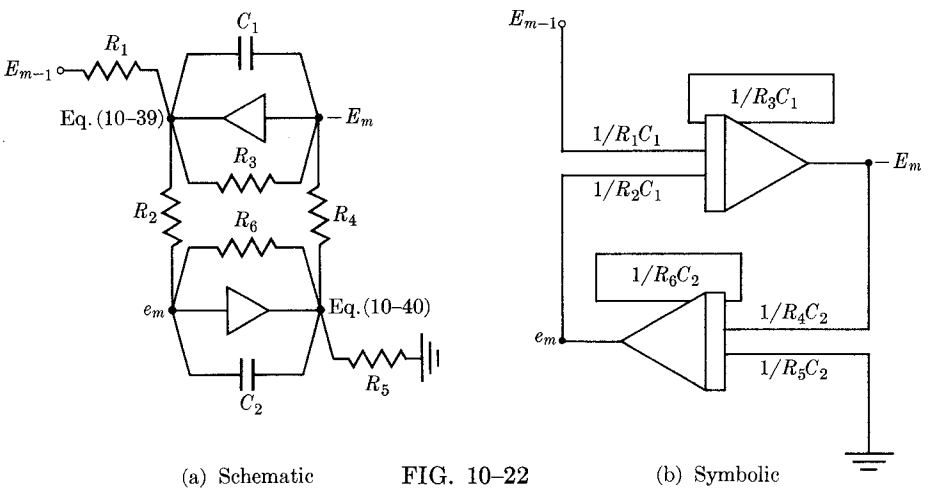
$$\frac{d\theta_m}{dt} = \frac{h_iP_i}{\rho'c'A'}T_m + \left(\frac{h_iP_i + h_oP_o}{\rho'c'A'}\right)(-\theta_m) + \frac{h_oP_o}{\rho'c'A'}0. \tag{10–38}$$

The zero ambient temperature is intentionally left in Eq. (10–38) to determine the ground connection of the corresponding analog circuit element. Simulating Eqs. (10–37) and (10–38), and reversing the signs of the second equation thus obtained, we get

$$\frac{E_{m-1}}{R_1} + \frac{e_m}{R_2} + \frac{(-E_m)}{R_3} + C_1\frac{d(-E_m)}{dt} = 0, \tag{10–39}$$

$$\frac{(-E_m)}{R_4} + \frac{(-0)}{R_5} + \frac{e_m}{R_6} + C_2\frac{de_m}{dt} = 0, \tag{10–40}$$

where the potentials E and e are used for the temperatures T and θ, respectively. Equations (10–39) and (10–40) are satisfied by the circuit element shown in Fig. 10–22. An appropriate number of such circuit elements should now be connected in series to cover the entire length of the tube. This, however, presents no problem and is left to the reader.



(a) Schematic FIG. 10–22 (b) Symbolic

10–5. Miscellaneous

In this chapter we have given a short introduction to the analog solution of conduction problems. In conclusion, we discuss briefly a number of important points related to mechanics of computer operation or to advanced topics.

Units. The units commonly employed in practice are the megohm (10^6 ohms) for resistors and the microfarad (10^{-6} farad) for capacitors. The product RC is then expressed in ohm $\times$ farad $=$ sec.

Amplitude and time scale factors. Variables of conduction and related computer problems have limitations on their possible magnitudes and rates of change with time; hence in practice the corresponding variables of the two systems must be related by amplitude and time scale factors. More specifically, an amplitude scale factor is needed because the computer variable, the output potential of a DC amplifier, is limited, often to ± 100 volts; in order to measure a temperature variation over a range greater than $\pm 100°$F, we would need a scale factor between potentials and temperatures. Similarly, the use of a time scale factor becomes necessary when the transient time of a problem is too short or too long; for short transients the components of the computer and of the recording equipment may not function adequately; for long transients amplifier drift and capacitor leakage may introduce appreciable errors. Furthermore, for reasons of economy, we often prefer to use the computer for a shorter time than the actual time. An extensive treatment of scale factors may be found in Chapter 3 of Reference 3.

Stability. When the gain through an amplifier is accompanied by a phase shift, the latter may lead to a condition of self-oscillation (instability) in the amplifier output. This important problem is beyond the scope of the text. A short discussion may be found in Section 7–8 of Reference 2.

High-speed computers. In a high-speed computer a problem is solved over and over again, many times in a second. Because of its limited component capability, this computer has a relatively low accuracy. Since no available graphical recording instrument is suitable to high-speed computers, the cathode-ray oscilloscope appears to be the only practical display device. Reference 7 is devoted to this type of computer.

Schematic and symbolic diagrams. In this chapter we have discussed circuit elements and corresponding circuits in terms of schematic and symbolic diagrams. The use of symbolic notation is desirable because of its simplicity; however, this notation often causes confusion for the beginner. Therefore, until enough experience has been gained in the use of symbolic diagrams, the use of schematic diagrams is suggested.

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Problems

10–1. Draw the passive circuit element corresponding to the conduction of heat governed by the equation

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial x} = a\frac{\partial^2 T}{\partial x^2}.$$

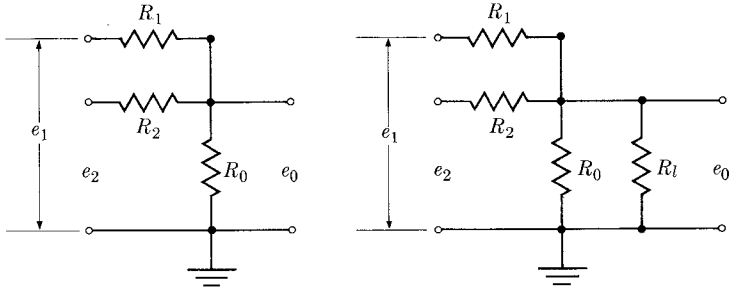
10–2. Draw the passive circuit elements corresponding to the boundary conditions

$$\pm k\left(\frac{\partial T}{\partial n}\right)_\sigma + q'' = h(T_\sigma - T_\infty)$$

and

$$\pm k_1\left(\frac{\partial T_1}{\partial n}\right)_\sigma + \mu pV = \pm k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma.$$

10–3. Find the effect of loading on the passive circuit element (adder) shown in Fig. 10–23.



(a) No load (b) With load

FIG. 10–23

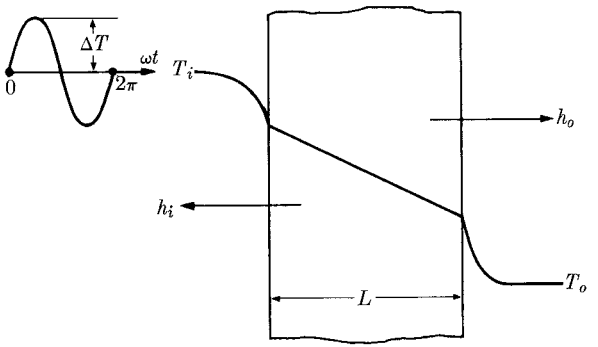


FIG. 10–24

10–4. Find the effect of loading on the passive circuit element (integrator) shown in Fig. 10–5.

10–5. Find the solution of Example 10–3 in terms of a passive circuit.

10–6. Find the solution of Example 2–2 in terms of an active circuit.

10–7. Draw the active circuit element for a typical inner nodal point of Problem 5–5. Connect three of these elements corresponding to three neighboring points.

10–8. Consider a wall through which heat is steadily transferred from an ambient of higher temperature to an ambient of lower temperature (Fig. 10–24). Assume that the higher ambient temperature starts to oscillate in the form

$$T_i(t) = T_i + \Delta T \sin \omega t.$$

(a) Using DC amplifiers, design a (function generator) circuit that generates sine wave fluctuations of the ambient temperature. (b) Give the solution of the problem in terms of an active circuit.

For the next three problems the reader is urged to use a locally available analog computer.

10–9. Solve Problem 9–9 by using an active circuit.

10–10. When axial conduction is included, Laplace transforms are no longer suitable to Problem 7–10. Investigate the effect of axial conduction on this problem by means of an active circuit.

10–11. Solve the general formulation of Example 2–5 in terms of an active circuit.

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